The Esoteric Keyboard Temperaments of J. S. Bach

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01 February 2005

In the year 1706, a young man, Johann George Neidhardt, full of importance as the author of a new book on temperament, held a tuning contest with Johann Sebastian's cousin, Johann Nikolaus Bach [1]. Neidhardt tuned one set of organ pipes using a monochord, while Bach tuned another entirely by ear. Johann Nikolaus won the contest, handily, for a singer found it easier to sing a chorale in Bb minor with his tuning. Some sixteen years later, Johann Sebastian, maintaining the family association with keyboard temperament, prepared the cover sheet for *Das wohltemperirte Clavier* [2]. This important work contained a Prelude and Fugue in every major and minor key, making a suitably tempered keyboard prerequisite for a complete performance. Bach commenced the title page of his work with the glyph shown above. To musicologists over the centuries, this was mere decoration, but at the close of the last millennium, the mathematician Andreas Sparschuh from the Technical University Darmstadt had a revelatory insight, proposing that the glyph depicted a sequence of numbers *1, 1, 1, 0, 0, 0, 2, 2, 2, 2, 2* representing coded instructions for tuning a keyboard [3].

Various temperaments concealed within the glyph are derived in this article under the assumption that the loops denote beats-rates per second for fifths on the circle-of-fifths (Figure 5). With regard to the historical measurement of time, the subdivision of hours into 60 minutes of 60 seconds was first proposed by medieval astronomers in the middle of 13th Century. During the 14th Century mechanical clocks, using weights and springs, began to appear. At first, they had no faces, and no hour or minute hands, but instead they struck a bell every hour. Subsequently, clocks with hour, and then minute hands began to appear and by the 15th Century there were small coiled springs unwinding at a speed controlled by an escapement, a discovery that made smaller clocks, and later watches, possible. Galileo Galilei is credited with inventing the pendulum-clock concept itself around 1582, sketching a

design for such a clock. In 1656, Christiaan Huygens made the first such pendulum clock, regulated by a mechanism with a natural period of oscillation. Huygens' early model had an error of less than 1 minute a day, the first time such accuracy had been achieved, but later refinements reduced the clock's error to less than 10 seconds a day. Eventually, the historical metre came to be defined as the length of a pendulum that beats once per second. Now, thanks to the discovery of the pendulum, anyone with a weight and a piece of cord could construct a device to count seconds.

Several pitch standards were in use in Bach's day. Most German organs made in his lifetime were at *Cornet-ton* pitch where *á* = 460-470 Hz with a mean of *á* = 465 Hz [4]. Another pitch standard used by Bach was a tone lower, *Cammerton (Kammerton)*, standardised today for the benefit of period-instrument performers at *á* =415 Hz; *tief-Cammerton* was one semitone lower. Bach's chamber and orchestral works were performed at Cammerton, but in a church setting 'figural' instruments (woodwinds, horns, and often strings) at (tief)-Cammerton could be used with an interval to Cornet-ton of either a major second or minor third. The simultaneous use of two pitch standards created the need to coordinate the intonation of the respective instruments.

Beat-rates for some example temperaments at Cornet-ton and Cammerton pitch standards are shown in Table 1. Note that for the pitches considered, the beat-rates in Equal Temperament do not exceed 2 beats-per-second, and that the fifths in Cornet-ton beat faster than Cammerton. Returning to Bach's glyph, the use of beat-rates 0, 1 and 2, represents an explicit decision to adopt an unequal beating temperament with the consequence that each key has its own colour. In this regard, both Neidhardt and Werckmeister explained that in their system the key of *C* should be the best and *Db* the worst, with the rest between the two extremes [1]. As indicated in Figures 1-4, however, the reality is somewhat more complicated[2].

Figure 6 illustrates the procedure for tuning a circle-of-fifths[3]. Tuning generally proceeds clockwise, adding sharps (e.g., C*:G, G:D,* etc.), with octave leaps downwards where appropriate. However, it is also possible to perform the reverse procedure by tuning fifths in the opposite direction (anti-clockwise), adding flats (e.g. *C:F, F:Bb,* etc.). There are, moreover, two ways of reading the glyph, left-to-right (clockwise) and right-to-left (anticlockwise), which taken together with the tuning direction just mentioned, yields four options. The four possibilities are mapped to the horizontal and vertical reflections of Bach's glyph in Figure 7.

[2] In interpreting these figures, note that a major or minor scale consists of the juxtaposition of two contiguous tetrachords on the circle-of-fifths. For example, the scale C Major consists of one tetrachord (C, D, E, F) followed by another (G, A, B, C). As a consequence, the "goodness" of any major or minor scale is primarily determined by the "goodness" of its two component tetrachords. Moreover, the "goodness" of each tetrachord can be expressed as the "distance" from the ideal pure tetrachord defined by selecting natural harmonics (overtones) of a fundamental. The "distance" is defined mathematically in terms of the four individual errors for each note in the tetrachord, combined using a 4-dimensional extension of Pythagoras Theorem by taking the square root of the sum of the squares of the four errors, a, so-called, Euclidian metric.

[3] While there are other procedures for tuning a keyboard using a circle-of-fifths, this is the method that tunes a range of contiguous semitones. An octave leap is made downwards whenever possible without going below the starting note. Without such octave leaps, the tuner will soon run out of keys, and, moreover, will have difficulty hearing the beat-rate of the fifths at higher frequencies. Another issue is that the beat-rate doubles at each octave, for a given tempering of fifth. Constraining the tuning to the smallest possible range of the keyboard is prerequisite for assigning a consistent interpretation to the numbers derived from the glyph.

As the tuning sequence can be started at any one of twelve positions on the glyph, there are 48 (i.e., 4 times 12) possibilities (Figure 8). There are 11 inner components to the glyph, while at the ends of the glyph, Bach indicates two different situations, with beat-rates of 1 and 2 respectively[4]. For the sake for completeness, the case of a pure end interval is also considered. In total, there are 144 options, each associated with a unique system of linear equations. Solving a particular system yields the specific frequencies for each semitone with an associated temperament. One representative system of equations is given in Equation 1. The complete set of solutions to all 144 systems is given in Tables 2 to 25. Analysis of results is restricted to 72 solutions, namely those corresponding to the top of Figure 7, where the tuning follows the circle-of-fifths in a clockwise direction by adding sharps[5].

As each solution equates to a specific pitch for the instrument, not all solutions will have had a value for Bach. Fortunately, we have some information concerning historical pitch standards, so the tuning options appropriate to Bach's time can be selected (Table 26). Consideration of the frequency information contained in Tables 2 to 25 reveals that certain tuning solutions are transpositions of others (Table 27). Of interest are solutions separated by a whole tone as these provide a means to tune keyboard instruments in Cornet-ton and Cammerton, such that they can be used together with perfect intonation. One such Cornet-ton/Cammerton solution exists reading the glyph left-to-right (Figure 9), while another exists reading the glyph right-to-left (Figure 10). Detailed instructions for tuning these methods are shown in Figure 12 and Figure 13.

An analysis of the quality of the major and minor tetrachords is given in Figures 14 to 21. From the figures, it can be seen that commencing the tuning sequence at consecutive points on the glyph, "rotates" the temperament so that the best tetrachords move to consecutive keys on the circle-of-fifths. An interval analysis for thirds and fifths for the selected temperaments is given in Tables 28 to 32, while a complete interval analysis is given in Tables 33 to 37. A comparison with historic temperaments in terms of Euclidian and correlation metrics is provided in Figures 22 to 33. Figure 34 shows a circle of fifths with the location of the best thirds indicated for the two Cammerton-Cornet-ton solutions addressed in this paper.

Noting that the major third on the tonic note (e.g., *C:E*) is also part of the triad of its relative minor (e.g., *A:C:E*), it can be seen that the quality of the major third impacts both the quality of tonic major and its relative minor; and, moreover, both share the same key signature. Accordingly, a count was made of the frequency with which Bach uses each key signature in Clavier and Organ works indicated in the *Bach Werke Verzeichnis* (Figure 35). The totals were then correlated with the size of the thirds in each major key for the temperaments derived from the glyph (Figure 36). This correlation procedure was also repeated for historic temperaments. The results indicate the strongest correlation with Temperament R1-0, a workable solution for historic Cammerton corresponding to the mid-point of Temperament R2-1 (Cammerton) and R12-2 (Corntet-ton). The correlation achieved with temperament R1-0 exceeded all historic temperaments; although a comparable correlation was achieved by the temperament of J. S. Bach's pupil, Kirnberger. There was no obvious

[4] Later it will be shown that these rates correspond to Cammerton and Cornet-ton, respectively.
[5] This restriction is made to keep the article to a reasonable length.

tendency for Bach to use flat key more often in organ works or sharp keys more often in clavier works.

Cents values for 72 of the temperaments derived from the glyph are given in Tables 38 to 40. In the authors opinion, the temperaments in Tables 39 and 40 are to be preferred. Musicians are invited to first try out the excellent Cammerton and Cornet-ton solutions before proceeding to the others. The author has spent many happy hours during 2004 exploring these wonderful temperaments in relation to the music of J. S. Bach, and anticipates that musicians and listeners will be similarly appreciative.

Cornet-ton	*Pythag-orean*	*Aron's Meantone (1/4-Comma)*	*Silbermann Meantone (1/6-Comma)*	*Equal Beating*	*12-Tone Equal Temp.*
C:G	*0.0*	*2.6*	*1.7*	*1.3*	*0.9*
G:D	*0.0*	*3.9*	*2.6*	*1.3*	*1.4*
D:A	*0.0*	*2.9*	*1.9*	*1.3*	*1.1*
A:E	*0.0*	*4.4*	*2.9*	*1.3*	*1.6*
E:B	*0.0*	*3.3*	*2.2*	*1.3*	*1.2*
B:F#	*0.0*	*4.9*	*3.3*	*1.3*	*1.8*
F#:C#	*0.0*	*3.7*	*2.4*	*1.3*	*1.3*
C#:G#	*0.0*	*2.7*	*1.8*	*1.3*	*1.0*
G#:Eb	*17.8*	*27.5 (wide)*	*12.2 (wide)*	*1.3*	*1.5*
Eb:Bb	*0.0*	*3.1*	*2.1*	*1.3*	*1.1*
Bb:F	*0.0*	*4.7*	*3.1*	*1.3*	*1.7*
F:C	*0.0*	*3.5*	*2.3*	*1.3*	*1.3*

Cam-merton	*Pythag-orean*	*Aron's Meantone (1/4-Comma)*	*Silbermann Meantone (1/6-Comma)*	*Equal Beating*	*12-Tone Equal Temp.*
C:G	*0.0*	*2.3*	*1.5*	*1.1*	*0.8*
G:D	*0.0*	*3.4*	*2.3*	*1.1*	*1.3*
D:A	*0.0*	*2.6*	*1.7*	*1.1*	*0.9*
A:E	*0.0*	*3.8*	*2.6*	*1.1*	*1.4*
E:B	*0.0*	*2.9*	*1.9*	*1.1*	*1.1*
B:F#	*0.0*	*4.3*	*2.9*	*1.1*	*1.6*
F#:C#	*0.0*	*3.2*	*2.2*	*1.1*	*1.2*
C#:G#	*0.0*	*2.4*	*1.6*	*1.1*	*0.9*
G#:Eb	*16.0*	*24.0 (wide)*	*10.8 (wide)*	*1.1*	*1.3*
Eb:Bb	*0.0*	*2.7*	*1.8*	*1.1*	*1.0*
Bb:F	*0.0*	*4.1*	*2.7*	*1.1*	*1.5*
F:C	*0.0*	*3.1*	*2.0*	*1.1*	*1.1*

Table 1: the beat-rates per second of the tempered fifths in Cammerton and Cornet-ton

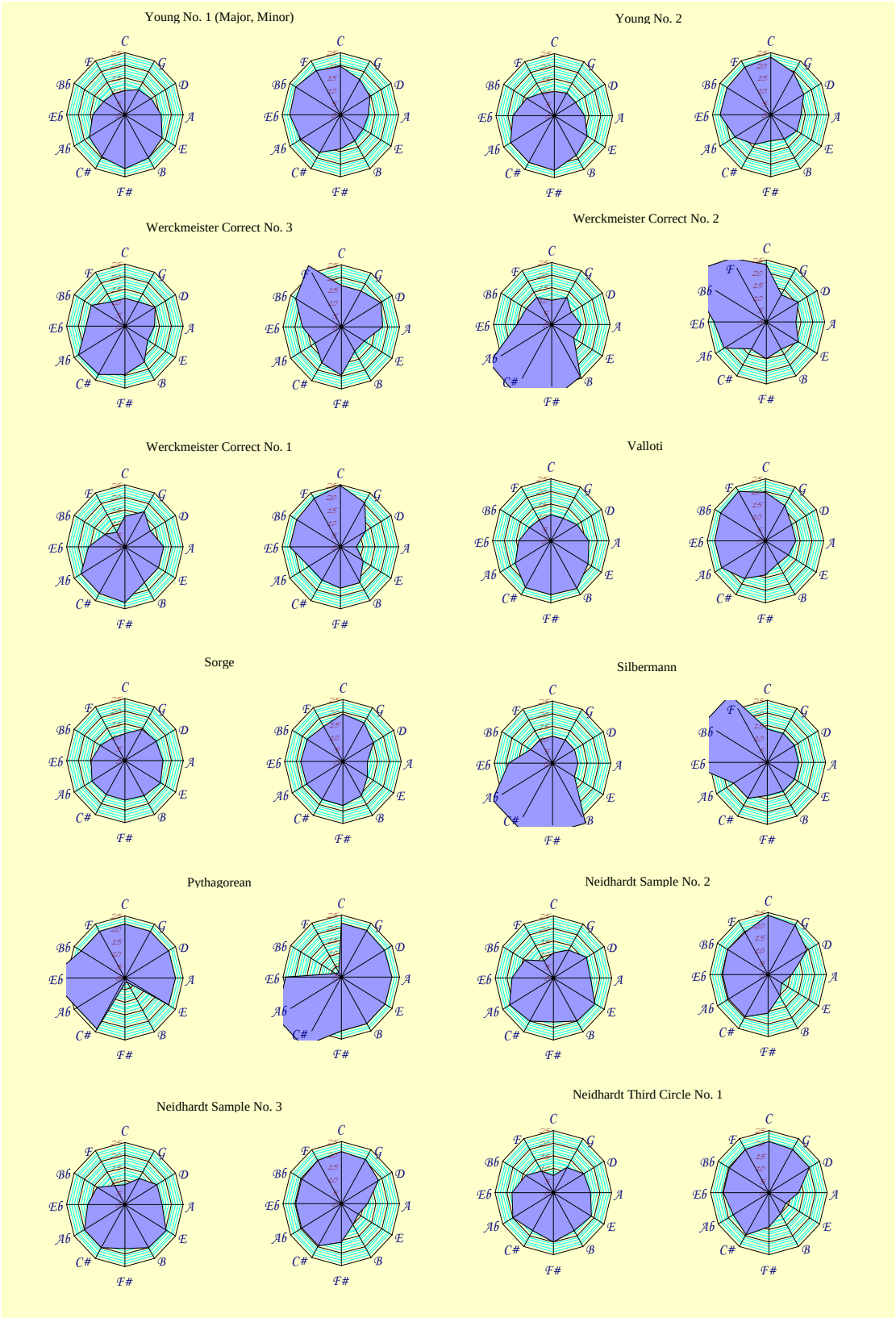


Figure 1: the quality of the major / minor tetrachords in historic temperaments (Euclidian distance in cents from the pure major / minor tetrachord) – I

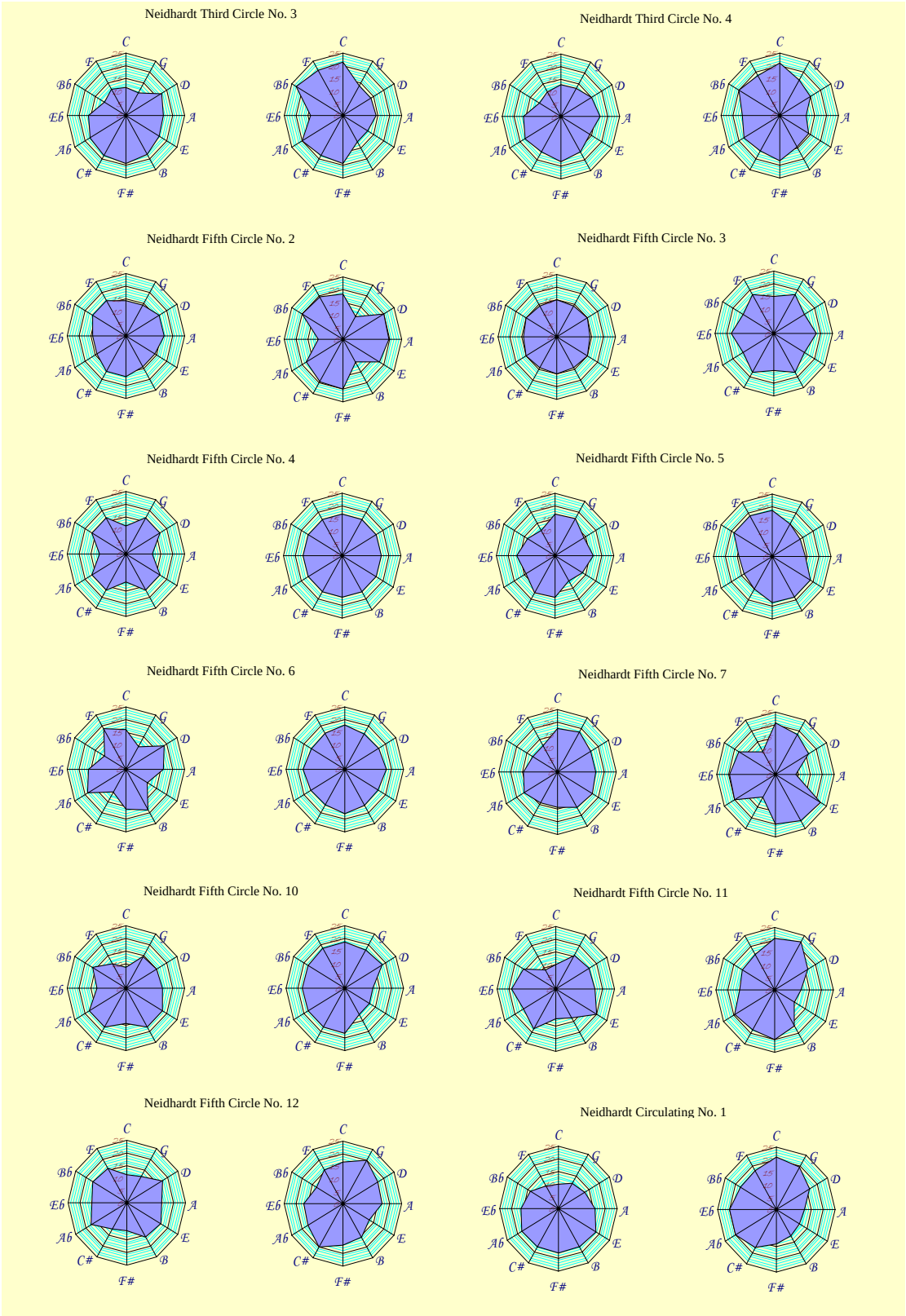


Figure 2: the quality of the major / minor tetrachords in historic temperaments (Euclidian distance in cents from the pure major / minor tetrachord) – II

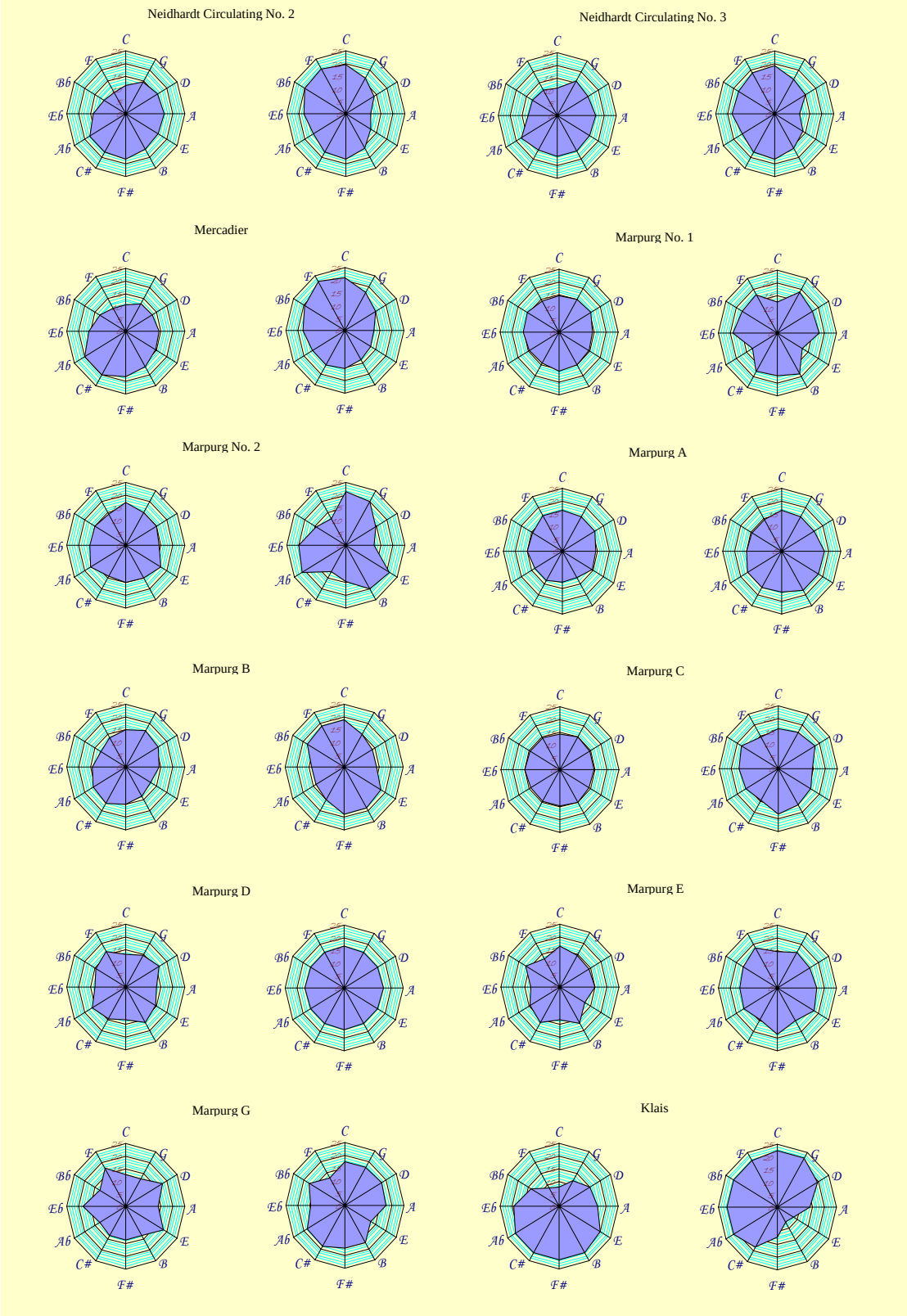


Figure 3: the quality of the major / minor tetrachords in historic temperaments (Euclidian distance in cents from the pure major / minor tetrachord) – III

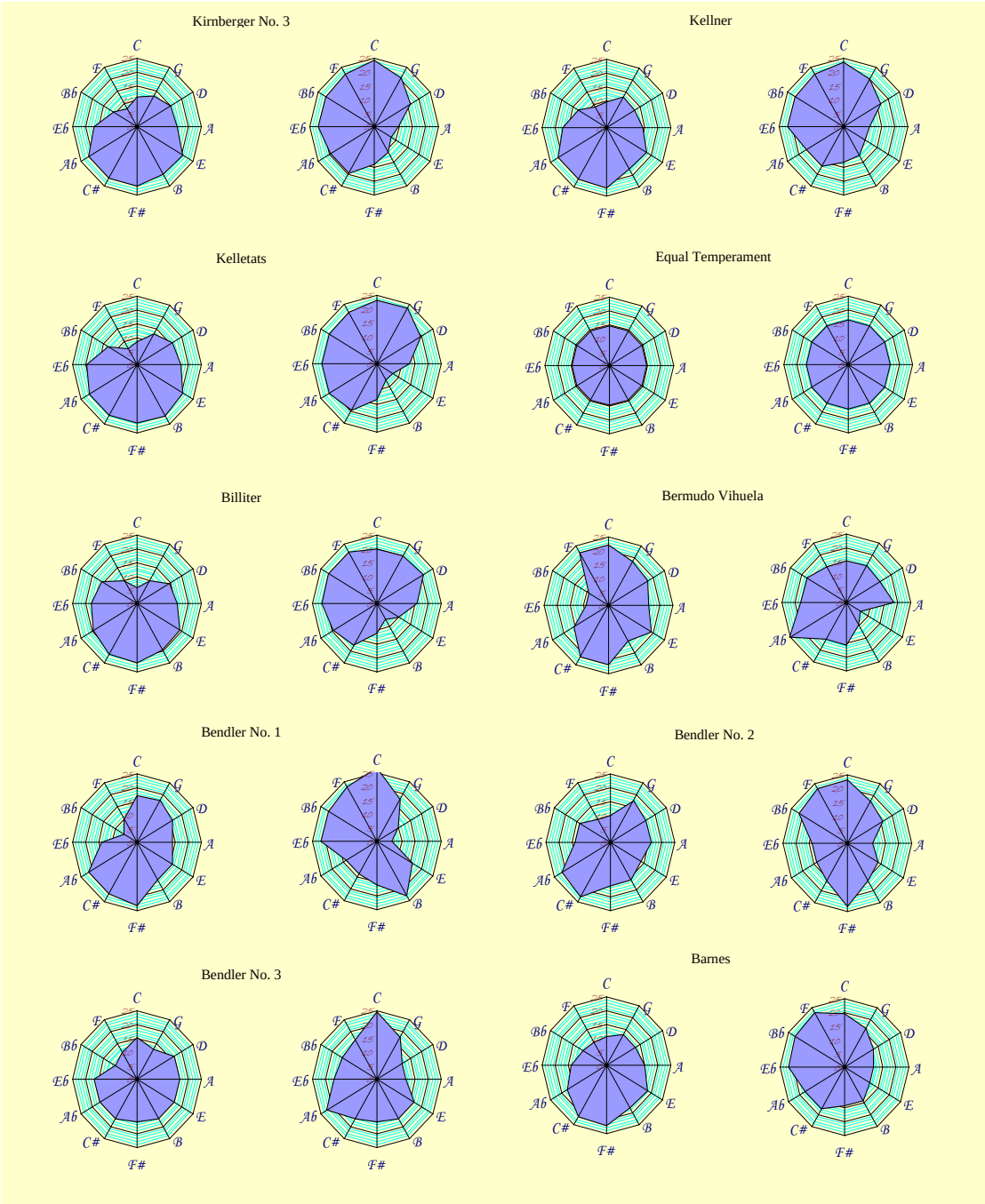


Figure 4: the best and worst major / minor tetrachords in historic temperaments (Euclidian distance in cents from the pure major / minor tetrachord) – IV

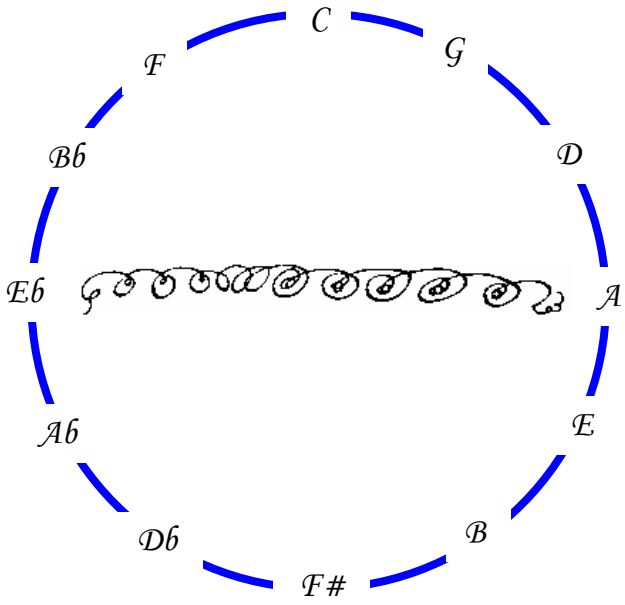


Figure 5: Bach's glyph encodes the properties of the circle-of-fifths in beat-rate per second

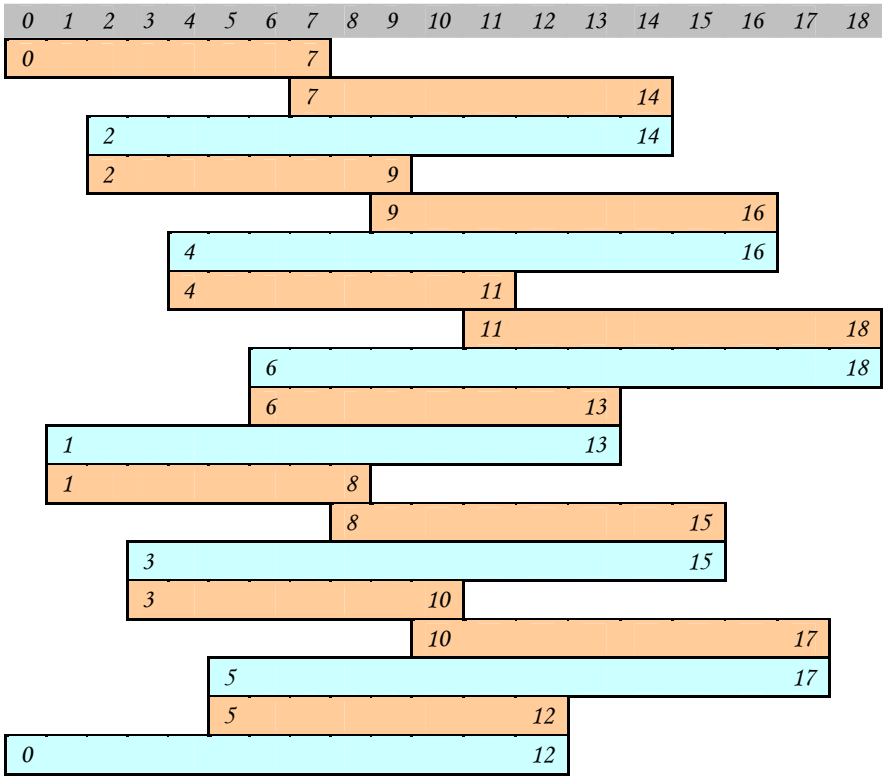


Figure 6: the procedure used for tuning contiguous semitones using a sequence of fifths on the circle-of-fifths with octave leaps. The corresponding beat-rates per second of each fifth on the circle-of-fifths are shown for the two cases of reading the glyph left to right and right to left. Typically, tuning proceed by adding sharps on the circle of fifth (e.g., C:G, G:D, etc.), however it is also possible to perform the reverse procedure by adding flats (e.g. C:F, F:Bb, etc.).

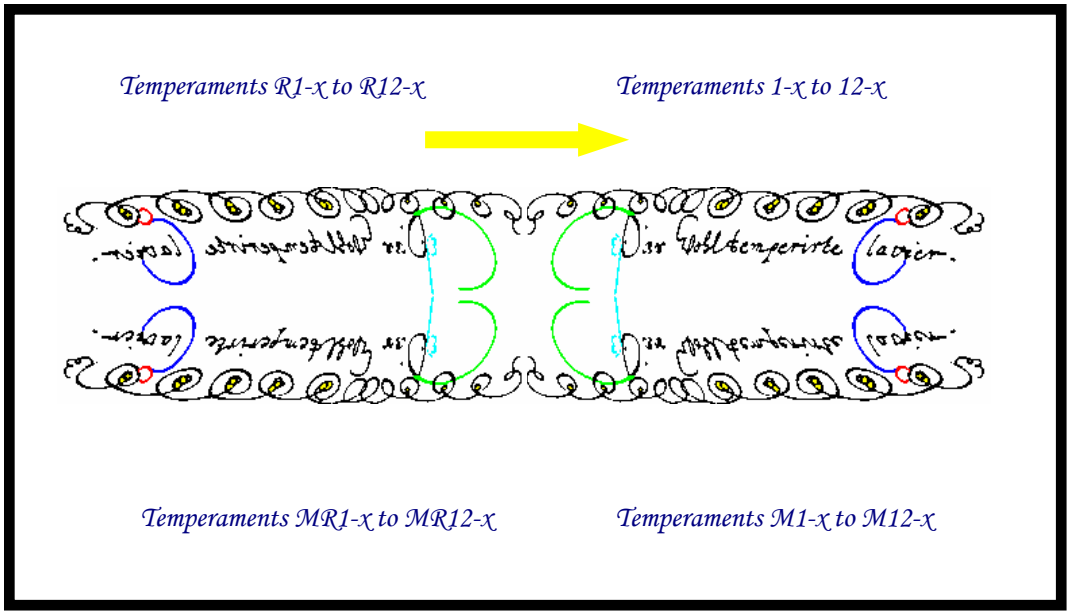


Figure 7: four classes of temperaments resulting from reading the glyph in two directions and proceeding in two directions along the circle-of-fifths mapped to the horizontal and vertical reflections of the glyph. The *D* in *Das* (top right) is combined with the mirror image of *d* (bottom right), while the "tail" of the *D* when reflected (bottom right) gives a *C* (compare to top right). The *b* (bottom left) forms part of the *B* arising from combining the top left image with the bottom right. The *C* of *Clavier* (top right) is joined to a smaller *c* that connects to a loop on the glyph.

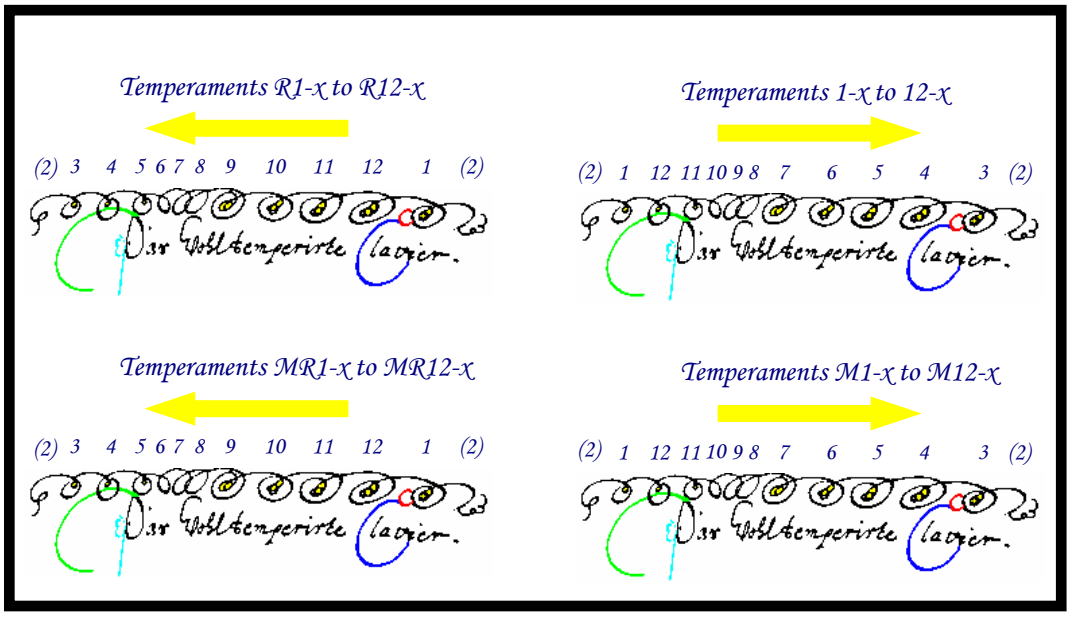
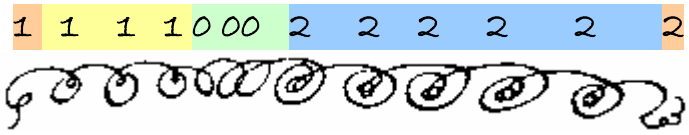


Figure 8: some 48-tuning options result from the glyph by starting the tuning procedure at any one of 12 positions on the glyph using four different methods. The methods at the top are based on tuning the circle-of-fifths in the direction of increasing sharps (e.g., C:G, G:D, etc.), while the lower ones tune the circle-of-fifths in the direction of increasing flats (e.g., C:F, F:Bb, etc.). The two methods on the right depict a reading of the glyph from left-to-right, while the two on the left denote a reading from right-to-left (reflection). Depending on where one starts on the glyph a different temperament results.



$3f_0 - 2f_7 = 1$	First loop in glyph
$3f_7 - 2f_{14} = 1$	Second loop in glyph
$2f_2 - f_{14} = 0$	Octave down
$3f_2 - 2f_9 = 1$	Third loop in glyph
$3f_9 - 2f_{16} = 0$	Fourth loop in glyph
$2f_4 - f_{16} = 0$	Octave down
$3f_4 - 2f_{11} = 0$	Fifth loop in glyph
$3f_{11} - 2f_{18} = 0$	Sixth loop in glyph
$2f_6 - f_{18} = 0$	Octave down
$3f_6 - 2f_{13} = 2$	Seventh loop in glyph
$2f_1 - f_{13} = 0$	Octave down
$3f_1 - 2f_8 = 2$	Eighths loop in glyph
$3f_8 - 2f_{15} = 2$	Ninth loop in glyph
$2f_3 - f_{15} = 0$	Octave down
$3f_3 - 2f_{10} = 2$	Tenth loop in glyph
$3f_{10} - 2f_{17} = 2$	Eleventh loop in glyph
$2f_5 - f_{17} = 0$	Octave down
$3f_5 - 2f_{12} = x$	End of glyph (x=0, x=1 or X=2)
$2f_0 - f_{12} = 0$	Octave down

Equation 1: the system of equations depicting Temperament 1 derived from the glyph by reading left-to-right starting at the far left loop. Each fifth consists of seven semitones and the beating results from interference between the third harmonic of lower note and the second harmonic of the higher one, where the beat-rate per second corresponds to the difference between these frequencies.

Table 2: Temperaments 1-0 to 6-0 (tuning circle-of-fifths toward sharps, glyph read left-right, end beat-rate 0)

Table 3: Temperaments 7-0 to 12-0 (tuning circle-of-fifths toward sharps, glyph read left-right, end beat-rate 0)

Semitone (Hz)	*1-1*	*2-1*	*3-1*	*4-1*	*5-1*	*6-1*
f_0	$\frac{1877093}{7153}$	$\frac{1864309}{7153}$	$\frac{1849648}{7153}$	$\frac{1839874}{7153}$	$\frac{1826842}{7153}$	$\frac{1901098}{7153}$
f_1	$\frac{1994410}{7153}$	$\frac{1982826}{7153}$	$\frac{1962612}{7153}$	$\frac{1949136}{7153}$	$\frac{1931168}{7153}$	$\frac{2008955}{7153}$
f_2	$\frac{2107259}{7153}$	$\frac{2092877}{7153}$	$\frac{2073701}{7153}$	$\frac{2060917}{7153}$	$\frac{2046256}{7153}$	$\frac{2129794}{7153}$
f_3	$\frac{2234770}{7153}$	$\frac{2221738}{7153}$	$\frac{2204362}{7153}$	$\frac{2192778}{7153}$	$\frac{2172564}{7153}$	$\frac{2257392}{7153}$
f_4	$\frac{2367984}{7153}$	$\frac{2350016}{7153}$	$\frac{2328443}{7153}$	$\frac{2314061}{7153}$	$\frac{2294885}{7153}$	$\frac{2387077}{7153}$
f_5	$\frac{2505175}{7153}$	$\frac{2490514}{7153}$	$\frac{2470966}{7153}$	$\frac{2457934}{7153}$	$\frac{2440558}{7153}$	$\frac{2539566}{7153}$
f_6	$\frac{2663982}{7153}$	$\frac{2643768}{7153}$	$\frac{2616816}{7153}$	$\frac{2598848}{7153}$	$\frac{2577275}{7153}$	$\frac{2680991}{7153}$
f_7	$\frac{2812063}{7153}$	$\frac{2792887}{7153}$	$\frac{2767319}{7153}$	$\frac{2752658}{7153}$	$\frac{2733110}{7153}$	$\frac{2844494}{7153}$
f_8	$\frac{2984462}{7153}$	$\frac{2967086}{7153}$	$\frac{2943918}{7153}$	$\frac{2923704}{7153}$	$\frac{2896752}{7153}$	$\frac{3009856}{7153}$
f_9	$\frac{3157312}{7153}$	$\frac{3135739}{7153}$	$\frac{3106975}{7153}$	$\frac{3087799}{7153}$	$\frac{3062231}{7153}$	$\frac{3187538}{7153}$
f_{10}	$\frac{3345002}{7153}$	$\frac{3325454}{7153}$	$\frac{3299390}{7153}$	$\frac{3282014}{7153}$	$\frac{3258846}{7153}$	$\frac{3386088}{7153}$
f_{11}	$\frac{3551976}{7153}$	$\frac{3525024}{7153}$	$\frac{3489088}{7153}$	$\frac{3467515}{7153}$	$\frac{3438751}{7153}$	$\frac{3577039}{7153}$

Table 4: Temperaments 1-1 to 6-1 (tuning circle-of-fifths toward sharps, glyph read left-right, end beat-rate 1)

Semitone (Hz)	*7-1*	*8-1*	*9-1*	*10-1*	*11-1*	*12-1*
f_0	$\frac{1889514}{7153}$	$\frac{1775988}{7153}$	$\frac{1762512}{7153}$	$\frac{1744544}{7153}$	$\frac{1822331}{7153}$	$\frac{1807949}{7153}$
f_1	$\frac{1994573}{7153}$	$\frac{1877093}{7153}$	$\frac{1864309}{7153}$	$\frac{1849648}{7153}$	$\frac{1933186}{7153}$	$\frac{1920154}{7153}$
f_2	$\frac{2116762}{7153}$	$\frac{1994410}{7153}$	$\frac{1982826}{7153}$	$\frac{1962612}{7153}$	$\frac{2047440}{7153}$	$\frac{2029472}{7153}$
f_3	$\frac{2239424}{7153}$	$\frac{2107259}{7153}$	$\frac{2092877}{7153}$	$\frac{2073701}{7153}$	$\frac{2165893}{7153}$	$\frac{2151232}{7153}$
f_4	$\frac{2372416}{7153}$	$\frac{2234770}{7153}$	$\frac{2221738}{7153}$	$\frac{2204362}{7153}$	$\frac{2303370}{7153}$	$\frac{2283156}{7153}$
f_5	$\frac{2519352}{7153}$	$\frac{2367984}{7153}$	$\frac{2350016}{7153}$	$\frac{2328443}{7153}$	$\frac{2432159}{7153}$	$\frac{2412983}{7153}$
f_6	$\frac{2661815}{7153}$	$\frac{2505175}{7153}$	$\frac{2490514}{7153}$	$\frac{2470966}{7153}$	$\frac{2582350}{7153}$	$\frac{2564974}{7153}$
f_7	$\frac{2827118}{7153}$	$\frac{2663982}{7153}$	$\frac{2643768}{7153}$	$\frac{2616816}{7153}$	$\frac{2729920}{7153}$	$\frac{2708347}{7153}$
f_8	$\frac{2988283}{7153}$	$\frac{2812063}{7153}$	$\frac{2792887}{7153}$	$\frac{2767319}{7153}$	$\frac{2892626}{7153}$	$\frac{2873078}{7153}$
f_9	$\frac{3167990}{7153}$	$\frac{2984462}{7153}$	$\frac{2967086}{7153}$	$\frac{2943918}{7153}$	$\frac{3071160}{7153}$	$\frac{3044208}{7153}$
f_{10}	$\frac{3359136}{7153}$	$\frac{3157312}{7153}$	$\frac{3135739}{7153}$	$\frac{3106975}{7153}$	$\frac{3245263}{7153}$	$\frac{3219695}{7153}$
f_{11}	$\frac{3551471}{7153}$	$\frac{3345002}{7153}$	$\frac{3325454}{7153}$	$\frac{3299390}{7153}$	$\frac{3447902}{7153}$	$\frac{3424734}{7153}$

Table 5: Temperaments 7-1 to 12-1 (tuning circle-of-fifths toward sharps, glyph read left-right, end beat-rate 1)

Semitone (Hz)	*1-2*	*2-2*	*3-2*	*4-2*	*5-2*	*6-2*
f_0	$\frac{2008165}{7153}$	$\frac{2041456}{7153}$	$\frac{1967746}{7153}$	$\frac{1997338}{7153}$	$\frac{1931818}{7153}$	$\frac{88742}{311}$
f_1	$\frac{2134378}{7153}$	$\frac{2169450}{7153}$	$\frac{2087028}{7153}$	$\frac{2115024}{7153}$	$\frac{2041760}{7153}$	$\frac{93757}{311}$
f_2	$\frac{2254715}{7153}$	$\frac{2289485}{7153}$	$\frac{2204773}{7153}$	$\frac{2238064}{7153}$	$\frac{2164354}{7153}$	$\frac{99446}{311}$
f_3	$\frac{2392234}{7153}$	$\frac{2431690}{7153}$	$\frac{2344330}{7153}$	$\frac{2379402}{7153}$	$\frac{2296980}{7153}$	$\frac{105360}{311}$
f_4	$\frac{2533872}{7153}$	$\frac{2571200}{7153}$	$\frac{2475899}{7153}$	$\frac{2510669}{7153}$	$\frac{2425957}{7153}$	$\frac{111488}{311}$
f_5	$\frac{2682322}{7153}$	$\frac{2726710}{7153}$	$\frac{2628430}{7153}$	$\frac{2667886}{7153}$	$\frac{2580526}{7153}$	$\frac{118530}{311}$
f_6	$\frac{2850606}{7153}$	$\frac{2892600}{7153}$	$\frac{2782704}{7153}$	$\frac{2820032}{7153}$	$\frac{2724731}{7153}$	$\frac{125113}{311}$
f_7	$\frac{3008671}{7153}$	$\frac{3055031}{7153}$	$\frac{2944466}{7153}$	$\frac{2988854}{7153}$	$\frac{2890574}{7153}$	$\frac{132802}{311}$
f_8	$\frac{3194414}{7153}$	$\frac{3247022}{7153}$	$\frac{3130542}{7153}$	$\frac{3172536}{7153}$	$\frac{3062640}{7153}$	$\frac{140480}{311}$
f_9	$\frac{3378496}{7153}$	$\frac{3430651}{7153}$	$\frac{3303583}{7153}$	$\frac{3349943}{7153}$	$\frac{3239378}{7153}$	$\frac{148858}{311}$
f_{10}	$\frac{3581198}{7153}$	$\frac{3640382}{7153}$	$\frac{3509342}{7153}$	$\frac{3561950}{7153}$	$\frac{3445470}{7153}$	$\frac{158040}{311}$
f_{11}	$\frac{3800808}{7153}$	$\frac{3856800}{7153}$	$\frac{3710272}{7153}$	$\frac{3762427}{7153}$	$\frac{3635359}{7153}$	$\frac{166921}{311}$

Table 6: Temperaments 1-2 to 6-2 (tuning circle-of-fifths toward sharps, glyph read left-right, end beat-rate 2)

Semitone (Hz)	*7-2*	*8-2*	*9-2*	*10-2*	*11-2*	*12-2*
f_0	1982826/7153	1900404/7153	1928400/7153	1855136/7153	1969787/7153	1906253/7153
f_1	2092877/7153	2008165/7153	2041456/7153	1967746/7153	2090650/7153	2025130/7153
f_2	2221738/7153	2134378/7153	**2169450/7153**	2087028/7153	2213328/7153	2140064/7153
f_3	2350016/7153	2254715/7153	**2289485/7153**	2204773/7153	2343040/7153	2269330/7153
f_4	2490514/7153	2392234/7153	**2431690/7153**	2344330/7153	2489994/7153	2407572/7153
f_5	2643768/7153	2533872/7153	**2571200/7153**	2475899/7153	2628767/7153	2544055/7153
f_6	2792887/7153	2682322/7153	**2726710/7153**	2628430/7153	2792302/7153	2704942/7153
f_7	2967086/7153	2850606/7153	**2892600/7153**	2782704/7153	2951104/7153	2855803/7153
f_8	3135739/7153	3008671/7153	**3055031/7153**	2944466/7153	3128822/7153	3030542/7153
f_9	3325454/7153	3194414/7153	**3247022/7153**	3130542/7153	3319992/7153	3210096/7153
f_{10}	3525024/7153	3378496/7153	**3430651/7153**	3303583/7153	3507407/7153	3396842/7153
f_{11}	3728618/7153	3581198/7153	**3640382/7153**	3509342/7153	3727838/7153	3611358/7153

Table 7: Temperaments 7-2 to 12-2 (tuning circle-of-fifths toward sharps, glyph read left-right, end beat-rate 2)

Table 8: Temperaments R1-0 to R6-0 (tuning circle-of-fifths toward sharps, glyph read right-left, end beat-rate 0)

Table 9: Temperaments R7-0 to R12-0 (tuning circle-of-fifths toward sharps, glyph read right-left, end beat-rate 0)

Semitone (Hz)	*R1-1*	*R2-1*	*R3-1*	*R4-1*	*R5-1*	*R6-1*
f_0	1882730/7153	1781615/7153	1764893/7153	1840901/7153	1826037/7153	1803834/7153
f_1	1990458/7153	1882344/7153	1862608/7153	1946038/7153	1931674/7153	1912522/7153
f_2	2109130/7153	1998058/7153	1981034/7153	2066543/7153	2049821/7153	2027525/7153
f_3	2237477/7153	2117637/7153	2095434/7153	2183928/7153	2164192/7153	2142646/7153
f_4	2363830/7153	2238874/7153	2219722/7153	2318602/7153	2301578/7153	2276495/7153
f_5	2512691/7153	2377871/7153	2355575/7153	2456919/7153	2434716/7153	2405112/7153
f_6	2653944/7153	2509792/7153	2488246/7153	2599486/7153	2580334/7153	2554798/7153
f_7	2816942/7153	2668846/7153	2643763/7153	2757775/7153	2735479/7153	2705751/7153
f_8	2985687/7153	2823516/7153	2793912/7153	2911904/7153	2890358/7153	2861630/7153
f_9	3156542/7153	**2989934/7153**	2964398/7153	3096238/7153	3071155/7153	3037711/7153
f_{10}	3352639/7153	**3172879/7153**	3143151/7153	3275892/7153	3246288/7153	3206816/7153
f_{11}	3538592/7153	**3351158/7153**	3322430/7153	3470750/7153	3445214/7153	3411166/7153

Table 10: Temperaments R1-1 to R6-1 (tuning circle-of-fifths toward sharps, glyph read right-left, end beat-rate 1)

Semitone (Hz)	*R7-1*	*R8-1*	*R9-1*	*R10-1*	*R11-1*	*R12-1*
f_0	$\frac{77784}{311}$	$\frac{1769296}{7153}$	$\frac{1852726}{7153}$	$\frac{1838362}{7153}$	$\frac{1912522}{7153}$	$\frac{82598}{311}$
f_1	$\frac{82598}{311}$	$\frac{1882730}{7153}$	$\frac{1968239}{7153}$	$\frac{1951517}{7153}$	$\frac{2027525}{7153}$	$\frac{87507}{311}$
f_2	$\frac{87507}{311}$	$\frac{1990458}{7153}$	$\frac{2078952}{7153}$	$\frac{2059216}{7153}$	$\frac{2142646}{7153}$	$\frac{92534}{311}$
f_3	$\frac{92534}{311}$	$\frac{2109130}{7153}$	$\frac{2208010}{7153}$	$\frac{2190986}{7153}$	$\frac{2276495}{7153}$	$\frac{98251}{311}$
f_4	$\frac{98251}{311}$	$\frac{2237477}{7153}$	$\frac{2338821}{7153}$	$\frac{2316618}{7153}$	$\frac{2405112}{7153}$	$\frac{103712}{311}$
f_5	$\frac{103712}{311}$	$\frac{2363830}{7153}$	$\frac{2475070}{7153}$	$\frac{2455918}{7153}$	$\frac{2554798}{7153}$	$\frac{110338}{311}$
f_6	$\frac{110338}{311}$	$\frac{2512691}{7153}$	$\frac{2626703}{7153}$	$\frac{2604407}{7153}$	$\frac{2705751}{7153}$	$\frac{116676}{311}$
f_7	$\frac{116676}{311}$	$\frac{2653944}{7153}$	$\frac{2771936}{7153}$	$\frac{2750390}{7153}$	$\frac{2861630}{7153}$	$\frac{123586}{311}$
f_8	$\frac{123586}{311}$	$\frac{2816942}{7153}$	$\frac{2948782}{7153}$	$\frac{2923699}{7153}$	$\frac{3037711}{7153}$	$\frac{131105}{311}$
f_9	$\frac{131105}{311}$	$\frac{2985687}{7153}$	$\frac{3118428}{7153}$	$\frac{3088824}{7153}$	$\frac{3206816}{7153}$	$\frac{138490}{311}$
f_{10}	$\frac{138490}{311}$	$\frac{3156542}{7153}$	$\frac{3304862}{7153}$	$\frac{3279326}{7153}$	$\frac{3411166}{7153}$	$\frac{147221}{311}$
f_{11}	$\frac{147221}{311}$	$\frac{3352639}{7153}$	$\frac{3504655}{7153}$	$\frac{3474927}{7153}$	$\frac{3607668}{7153}$	$\frac{155568}{311}$

Table 11: Temperaments R7-1 to R12-1 (tuning circle-of-fiths toward sharps, glyph read right-left, end beat-rate 1)

Semitone (Hz)	*R1-2*	*R2-2*	*R3-2*	*R4-2*	*R5-2*	*R6-2*
f_0	$\frac{2013802}{7153}$	$\frac{1958762}{7153}$	$\frac{1882991}{7153}$	$\frac{1998365}{7153}$	$\frac{1931013}{7153}$	$\frac{1943802}{7153}$
f_1	$\frac{2130426}{7153}$	$\frac{2068968}{7153}$	$\frac{1987024}{7153}$	$\frac{2111926}{7153}$	$\frac{2042266}{7153}$	$\frac{2059978}{7153}$
f_2	$\frac{2256586}{7153}$	$\frac{2194666}{7153}$	$\frac{2112106}{7153}$	$\frac{2243690}{7153}$	$\frac{2167919}{7153}$	$\frac{2184989}{7153}$
f_3	$\frac{2394941}{7153}$	$\frac{2327589}{7153}$	$\frac{2235402}{7153}$	$\frac{2370552}{7153}$	$\frac{2288608}{7153}$	$\frac{2308534}{7153}$
f_4	$\frac{2529718}{7153}$	$\frac{2460058}{7153}$	$\frac{2367178}{7153}$	$\frac{2515210}{7153}$	$\frac{2432650}{7153}$	$\frac{2453642}{7153}$
f_5	$\frac{2689838}{7153}$	$\frac{2614067}{7153}$	$\frac{2513039}{7153}$	$\frac{2666871}{7153}$	$\frac{2574684}{7153}$	$\frac{2591736}{7153}$
f_6	$\frac{2840568}{7153}$	$\frac{2758624}{7153}$	$\frac{2654134}{7153}$	$\frac{2820670}{7153}$	$\frac{2727790}{7153}$	$\frac{2751406}{7153}$
f_7	$\frac{3013550}{7153}$	$\frac{2930990}{7153}$	$\frac{2820910}{7153}$	$\frac{2993971}{7153}$	$\frac{2892943}{7153}$	$\frac{2915703}{7153}$
f_8	$\frac{3195639}{7153}$	$\frac{3103452}{7153}$	$\frac{2980536}{7153}$	$\frac{3160736}{7153}$	$\frac{3056246}{7153}$	$\frac{3082814}{7153}$
f_9	$\frac{3377726}{7153}$	$\frac{3284846}{7153}$	$\frac{3161006}{7153}$	$\frac{3358382}{7153}$	$\frac{3248302}{7153}$	$\frac{3273907}{7153}$
f_{10}	$\frac{3588835}{7153}$	$\frac{3487807}{7153}$	$\frac{3353103}{7153}$	$\frac{3555828}{7153}$	$\frac{3432912}{7153}$	$\frac{3455648}{7153}$
f_{11}	$\frac{3787424}{7153}$	$\frac{3682934}{7153}$	$\frac{3543614}{7153}$	$\frac{3765662}{7153}$	$\frac{3641822}{7153}$	$\frac{3673310}{7153}$

Table 12: Temperaments R1-2 to R6-2 (tuning circle-of-fifths toward sharps, glyph read right-left, end beat-rate 2)

Semitone (Hz)	R7-2	R8-2	R9--2	R10-2	R11-2	R12-2
f_0	$\frac{1882344}{7153}$	$\frac{1893712}{7153}$	$\frac{2018614}{7153}$	$\frac{1948954}{7153}$	$\frac{2059978}{7153}$	$\frac{1998058}{7153}$
f_1	$\frac{1998058}{7153}$	$\frac{2013802}{7153}$	$\frac{2145386}{7153}$	$\frac{2069615}{7153}$	$\frac{2184989}{7153}$	$\frac{2117637}{7153}$
f_2	$\frac{2117637}{7153}$	$\frac{2130426}{7153}$	$\frac{2265576}{7153}$	$\frac{2183632}{7153}$	$\frac{2308534}{7153}$	$\frac{2238874}{7153}$
f_3	$\frac{2238874}{7153}$	$\frac{2256586}{7153}$	$\frac{2404618}{7153}$	$\frac{2322058}{7153}$	$\frac{2453642}{7153}$	$\frac{2377871}{7153}$
f_4	$\frac{2377871}{7153}$	$\frac{2394941}{7153}$	$\frac{2548773}{7153}$	$\frac{2456586}{7153}$	$\frac{2591736}{7153}$	$\frac{2509792}{7153}$
f_5	$\frac{2509792}{7153}$	$\frac{2529718}{7153}$	$\frac{2696254}{7153}$	$\frac{2603374}{7153}$	$\frac{2751406}{7153}$	$\frac{2668846}{7153}$
f_6	$\frac{2668846}{7153}$	$\frac{2689838}{7153}$	$\frac{2862899}{7153}$	$\frac{2761871}{7153}$	$\frac{2915703}{7153}$	$\frac{2823516}{7153}$
f_7	$\frac{2823516}{7153}$	$\frac{2840568}{7153}$	$\frac{3020768}{7153}$	$\frac{2916278}{7153}$	$\frac{3082814}{7153}$	$\frac{2989934}{7153}$
f_8	$\frac{2989934}{7153}$	$\frac{3013550}{7153}$	$\frac{3210926}{7153}$	$\frac{3100846}{7153}$	$\frac{3273907}{7153}$	$\frac{3172879}{7153}$
f_9	$\frac{3172879}{7153}$	$\frac{3195639}{7153}$	$\frac{3398364}{7153}$	$\frac{3275448}{7153}$	$\frac{3455648}{7153}$	$\frac{3351158}{7153}$
f_{10}	$\frac{3351158}{7153}$	$\frac{3377726}{7153}$	$\frac{3599774}{7153}$	$\frac{3475934}{7153}$	$\frac{3673310}{7153}$	$\frac{3563230}{7153}$
f_{11}	$\frac{3563230}{7153}$	$\frac{3588835}{7153}$	$\frac{3819583}{7153}$	$\frac{3684879}{7153}$	$\frac{3887604}{7153}$	$\frac{3764688}{7153}$

Table 13: Temperaments R7-2 to R12-2 (tuning circle-of-fifths toward sharps, glyph read right-left, end beat-rate 2)

Table 14: Temperaments M1-0 to M6-0 (tuning circle-of-fifths toward flats, glyph read left-right, end beat-rate 0)

Table 15: Temperaments M7-0 to M12-0 (tuning circle-of-fifths toward flats, glyph read left-right, end beat-rate 0)

Table 16: Temperaments M1-1 to M6-1 (tuning circle-of-fifths toward flats, glyph read left-right, end beat-rate 1)

Table 17: Temperaments M7-1 to M12-1 (tuning circle-of-fifths toward flats, glyph read left-right, end beat-rate 1)

Table 18: Temperaments M1-2 to M6-2 (tuning circle-of-fifths toward flats, glyph read left-right, end beat-rate 2)

Table 19: Temperaments M7-2 to M12-2 (tuning circle-of-fifths toward flats, glyph read left-right, end beat-rate 2)

Table 20: Temperaments MR1-0 to MR6-0 (tuning circle-of-fifths toward flats, glyph read right-left, end beat-rate 0)

Table 21: Temperaments MR7-0 to MR12-0 (tuning circle-of-fifths toward flats, glyph read right-left, end beat-rate 0)

Table 22: Temperaments MR1-1 to MR6-1 (tuning circle-of-fifths toward flats, glyph read right-left, end beat-rate 1)

Table 23: Temperaments MR7-1 to MR12-1 (tuning circle-of-fifths toward flats, glyph read right-left, end beat-rate 1)

Table 24: Temperaments MR1-2 to MR6-2 (tuning circle-of-fifths toward flats, glyph read right-left, end beat-rate 2)

Table 25: Temperaments MR7-2 to MR12-2 (tuning circle-of-fifths toward flats, glyph read right-left, end beat-rate 2)

Temperament	f_9 (Hz)	*Temperament*	f_9 (Hz)
1-0	410.475	*R1-0*	410.367
2-0	397.152	*R2-0*	376.768
3-0	406.874	*R3-0*	386.941
4-0	395.031	*R4-0*	396.211
5-0	403.339	*R5-0*	404.587
6-0	412.602	*R6-0*	391.656
7-0	420.876	*R7-0*	399.546
8-0	387.881	*R8-0*	388.052
9-0	375.668	*R9-0*	396.825
10-0	385.474	*R10-0*	405.732
11-0	394.566	*R11-0*	413.531
12-0	402.393	*R12-0*	422.114
1-1	441.397	*R1-1*	441.289
2-1	438.381	*R2-1*	417.997
3-1	434.36	*R3-1*	414.427
4-1	431.679	*R4-1*	432.859
5-1	428.104	*R5-1*	429.352
6-1	445.623	*R6-1*	424.676
7-1	442.89	*R7-1*	421.559
8-1	417.232	*R8-1*	417.403
9-1	414.803	*R9-1*	435.961
10-1	411.564	*R10-1*	431.822
11-1	429.353	*R11-1*	448.318
12-1	425.585	*R12-1*	445.305
1-2	472.319	*R1-2*	472.211
2-2	479.61	*R2-2*	459.226
3-2	461.846	*R3-2*	441.913
4-2	468.327	*R4-2*	469.507
5-2	452.87	*R5-2*	454.117
6-2	478.643	*R6-2*	457.697
7-2	464.903	*R7-2*	443.573
8-2	446.584	*R8-2*	446.755
9-2	453.938	*R9-2*	475.096
10-2	437.654	*R10-2*	457.912
11-2	464.14	*R11-2*	483.105
12-2	448.776	*R12-2*	468.497

Table 26: the frequency in Hz of f_9 can be compared with the estimated Cornet-ton pitch range *á* = 460-470 Hz (with a mean of 465 Hz) and the mean Cammerton value of *á* is 415 Hz. Temperaments within 5 Hz of the mean Cornet-ton and Cammerton values are highlighted.

f_0 of 1-1 = f_1 of 8-1	Relation
f_0 of MR2-2 = f_2 of MR12-1 = f_6 of 7-2 = f_7 of 2-1 = f_8 of 9-1	Tone relation (Cornet-ton–Cammerton) Tone relation (Cornet-ton–Cammerton)
f_0 of 3-1 = f_1 of 10-1	Semitone relation
f_0 of 1-2 = f_1 of 8-2	Semitone relation
f_0 of 2-2 = f_1 of 9-2	Semitone relation
f_0 of 3-2 = f_1 of 10-2	Semitone relation
f_0 of R1-1 = f_1 of R8-1	Semitone relation
f_0 of M9-2 = f_1 of M2-2 = f_6 of R12-2 = f_7 of R7-2 = f_8 of R2-1	Tone relation (Cornet-ton–Cammerton)
f_0 of R11-1 = f_1 of R6-1	Semitone relation
f_0 of R12-1 = f_1 of R7-1	Semitone relation
f_0 of R1-2 = f_1 of R8-2	Semitone relation
f_0 of R11-2 = f_1 of R6-2	Semitone relation

Table 27: the tuning solutions in this table are transpositions with a frequency relation as shown. The analysis is focused on Temperaments 1-x to 12-x and R1-x to R12-x. Temperaments M1-x to M12-x and MR1-x to MR12-x are not included in the tables apart from situations where they have an equivalence to those under consideration.

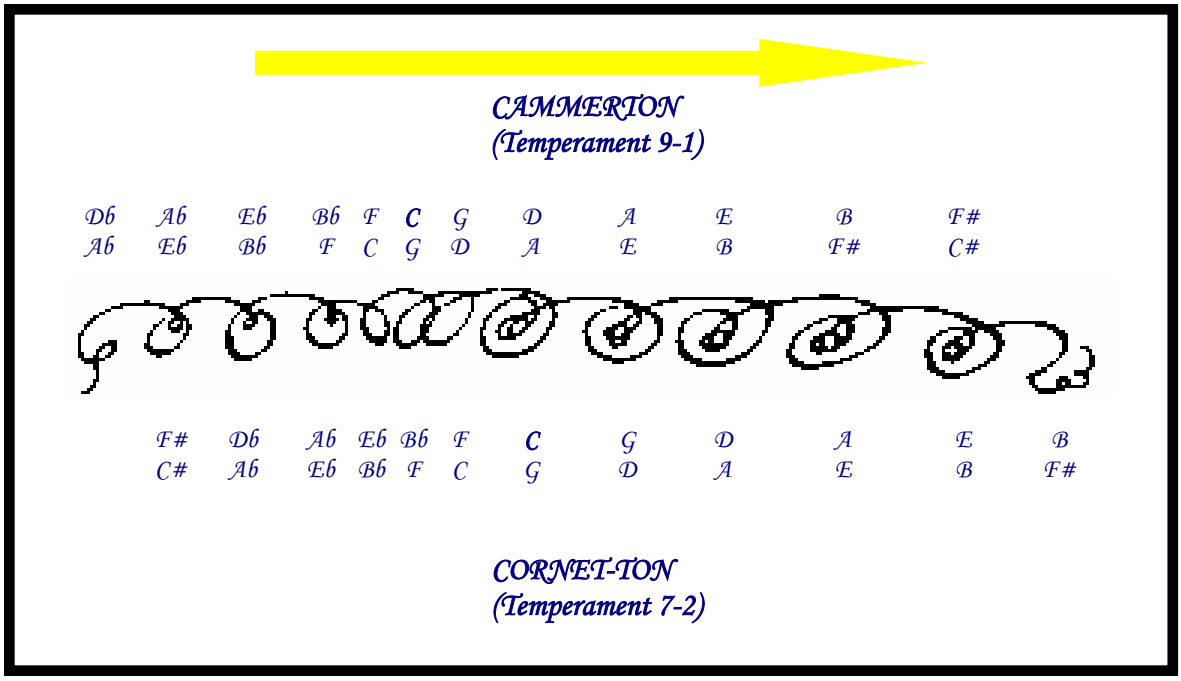


Figure 9: the correspondence between Temperaments 9-1 (Cammerton) and its transposition 7-2 (Cornet-ton) in relation to Bach's glyph. Each temperament is tuned by setting the beat-rate of the successive fifths on the circle-of-fifths according to number of small loops (0, 1, 2) by starting on C at the indicated point and reading the glyph left to right (clockwise). For Cammerton pitch, the left end of the glyph is considered, while for Cornet-ton the right end is used.

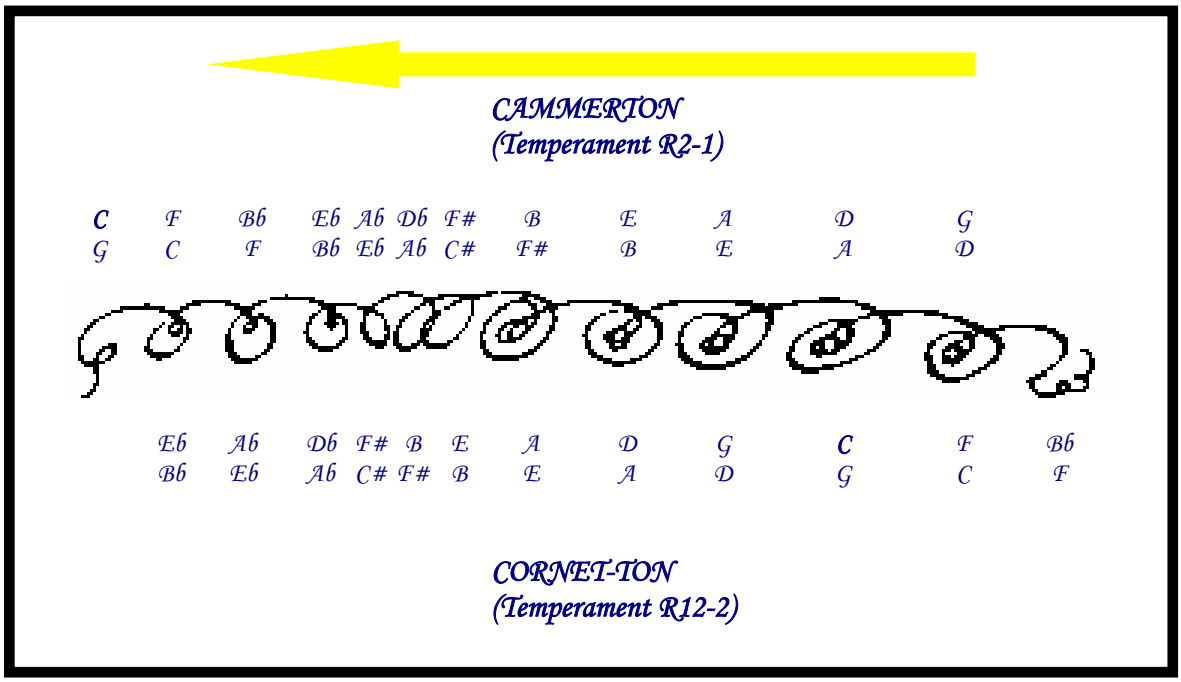


Figure 10: the correspondence between Temperaments R2-1 (Cammerton) and its transposition R12-2 (Cornet-ton) in relation to Bach's glyph. Each temperament is tuned by setting the beat-rate of the successive fifth on the circle-of-fifths according to number of small loops (0, 1, 2) by starting on C and reading the glyph right to left (anti-clockwise). For Cammerton pitch, the left end of the glyph is considered, while for Cornet-ton the right end is used.

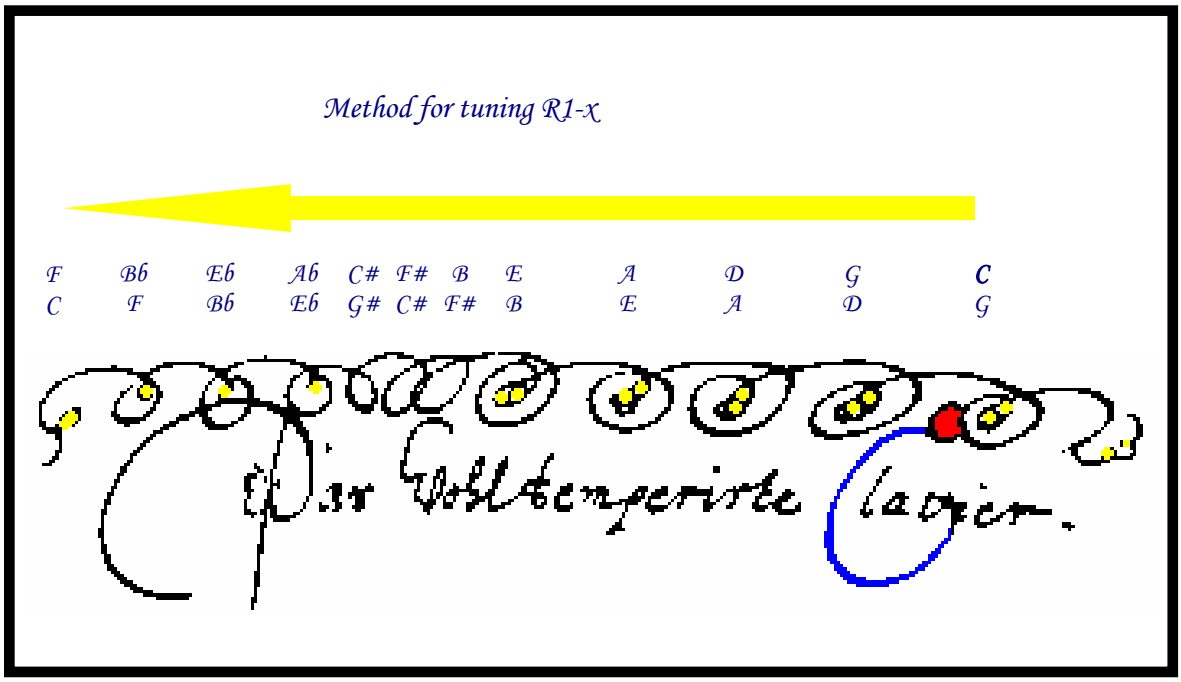


Figure 11: Temperaments R2-1 (Cammerton) and R12-2 (Cornet-ton) with best major third in C

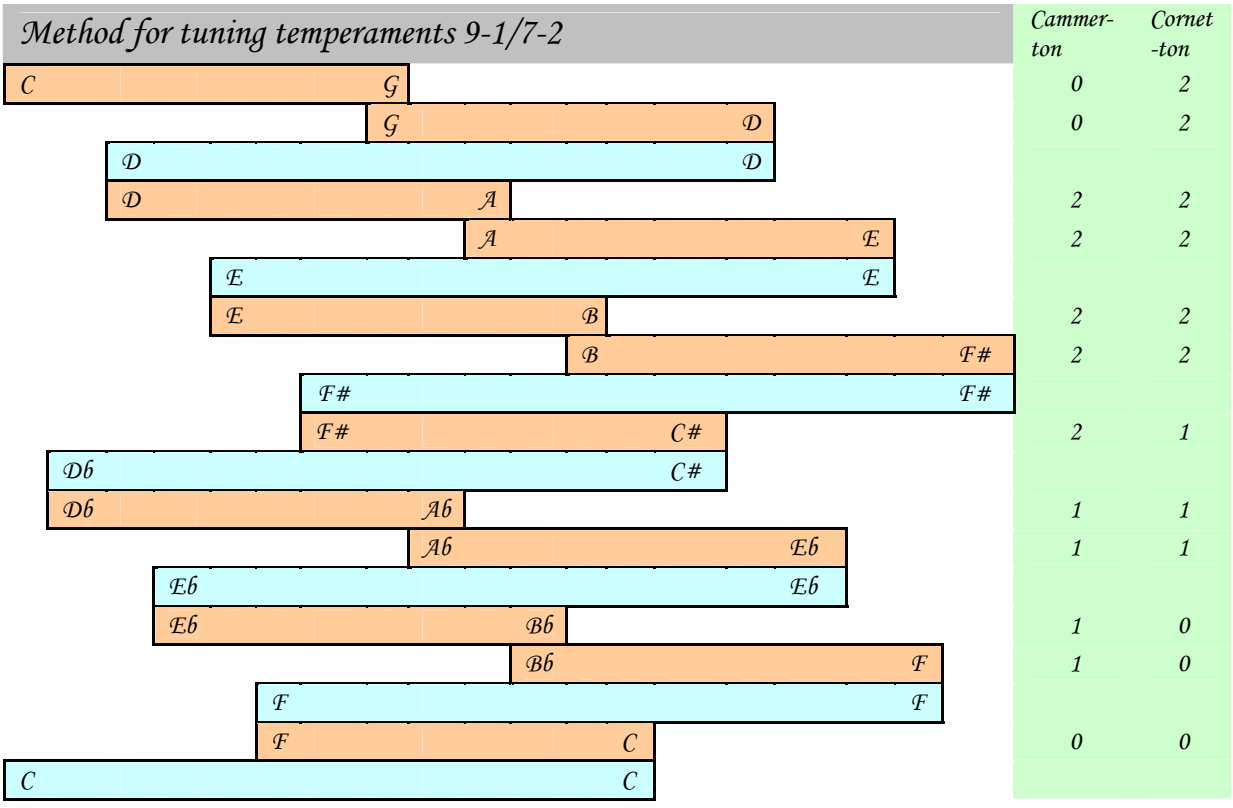


Figure 12: detailed tuning procedure for Temperaments 9-1 (Cammerton, *a* = 414.803 Hz) and 7-2 (Cornet-ton, *a* = 464.903 Hz)

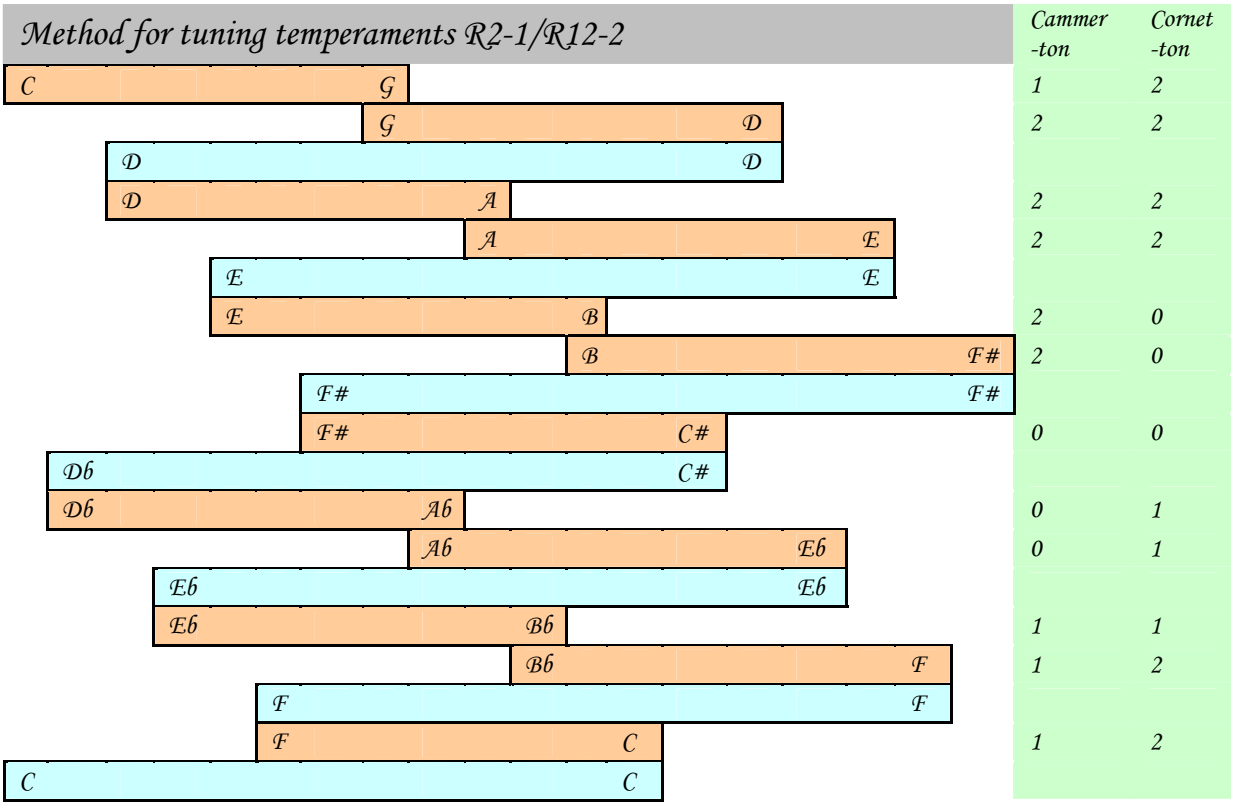


Figure 13: detailed tuning procedure for Temperaments R2-1 (Cammerton, *a* = 417.997 Hz) and R12-2 (Cornet-ton, *a* = 468.497 Hz)

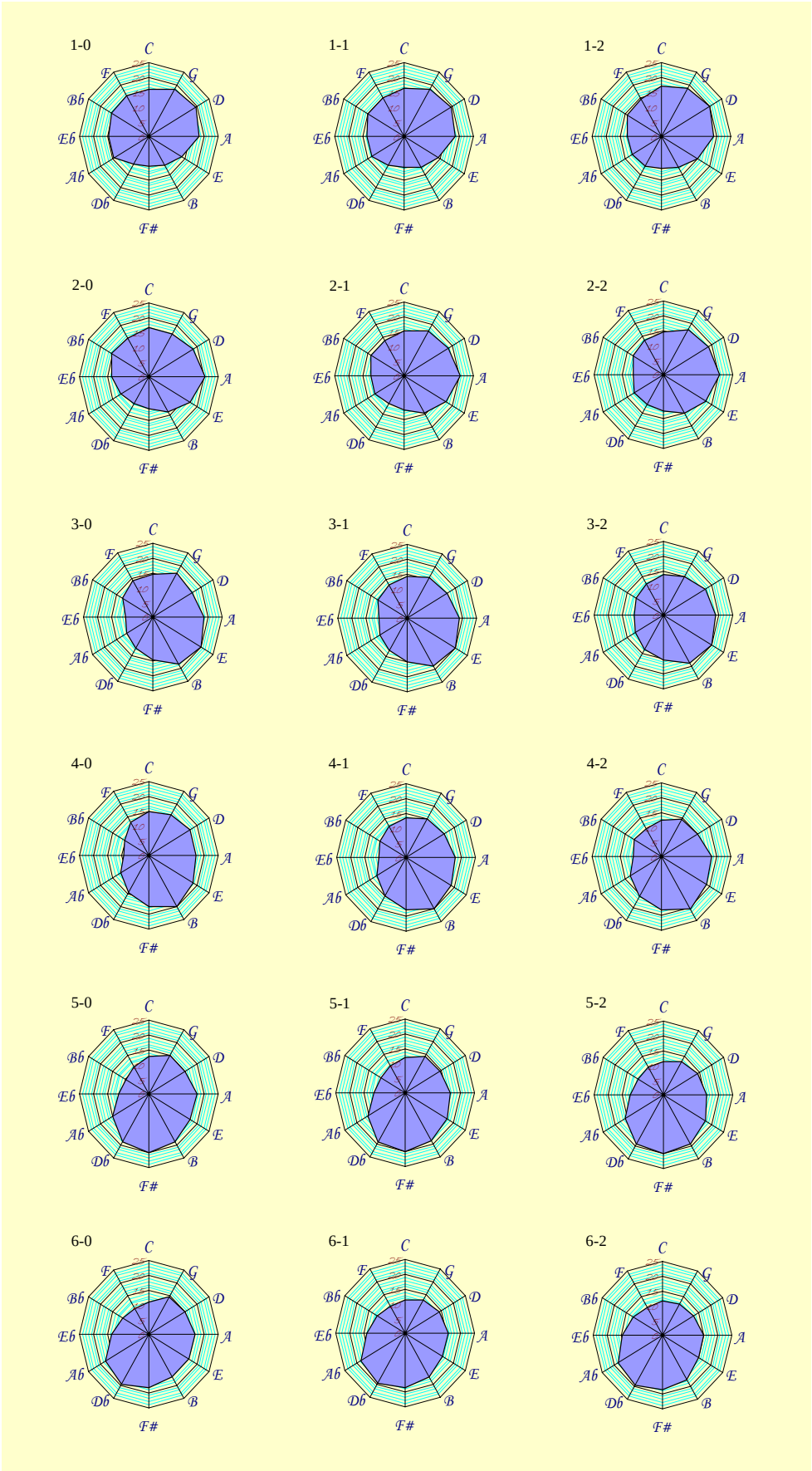


Figure 14: Euclidian distance in cents from the pure major tetrachord of each tetrachord in Temperaments 1-x to 6-x for end beat-rates of 0, 1 and 2

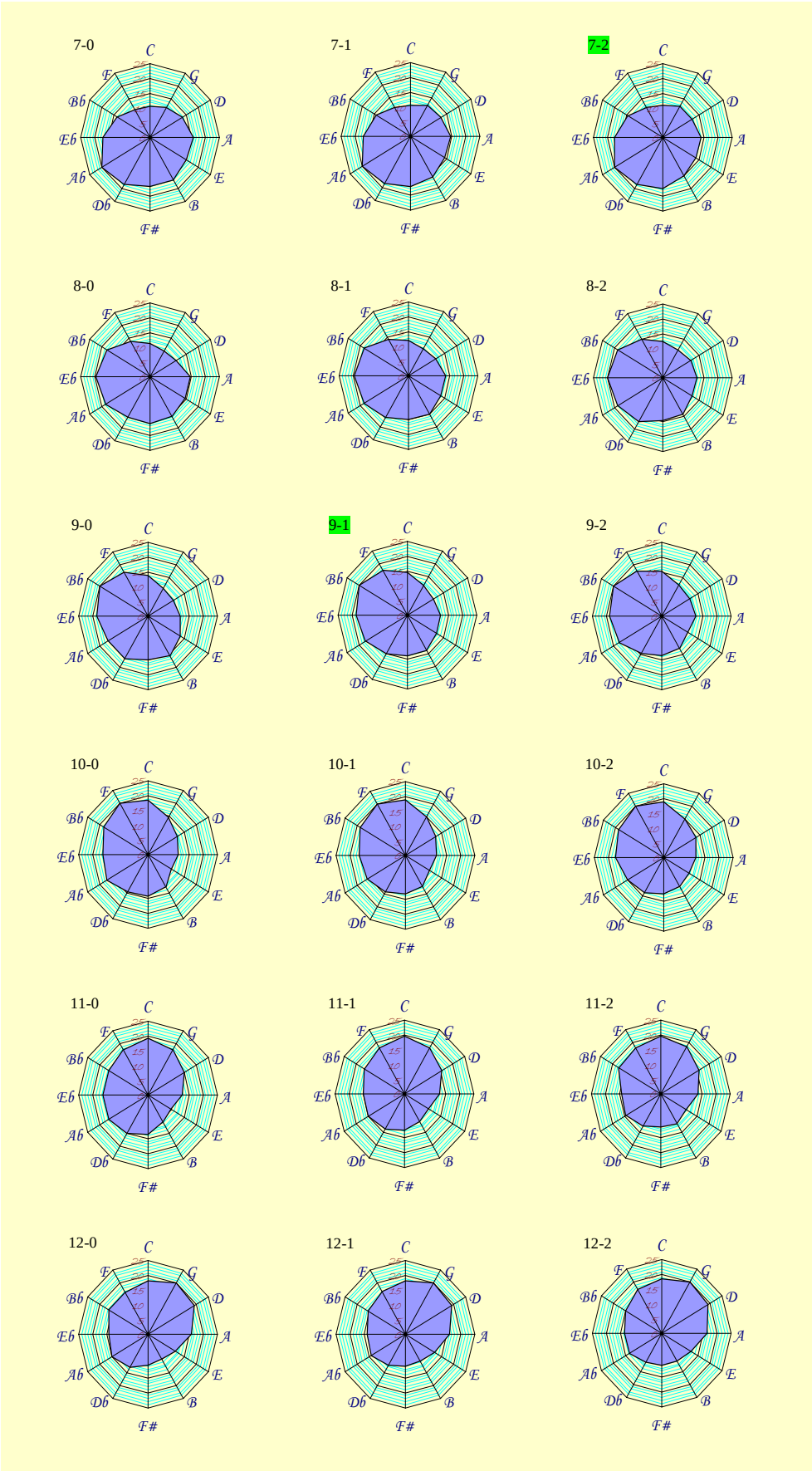


Figure 15: Euclidian distance in cents from the pure major tetrachord of each tetrachord in Temperaments 7-x to 12-x for end beat-rates of 0, 1 and 2

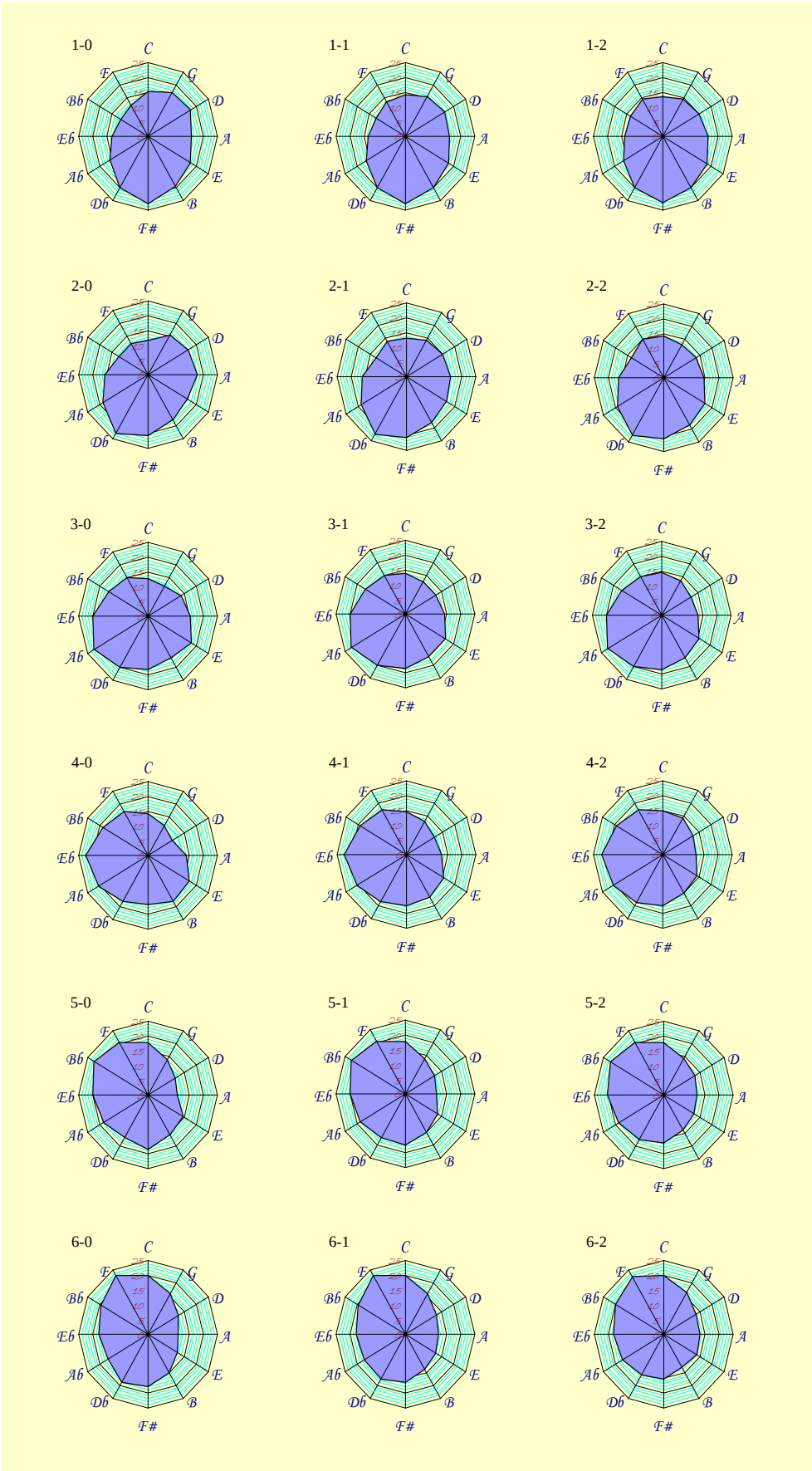


Figure 16: Euclidian distance in cents of the pure minor tetrachord from each tetrachord in Temperaments 1-x to 6-x for end beat-rates of 0, 1 and 2

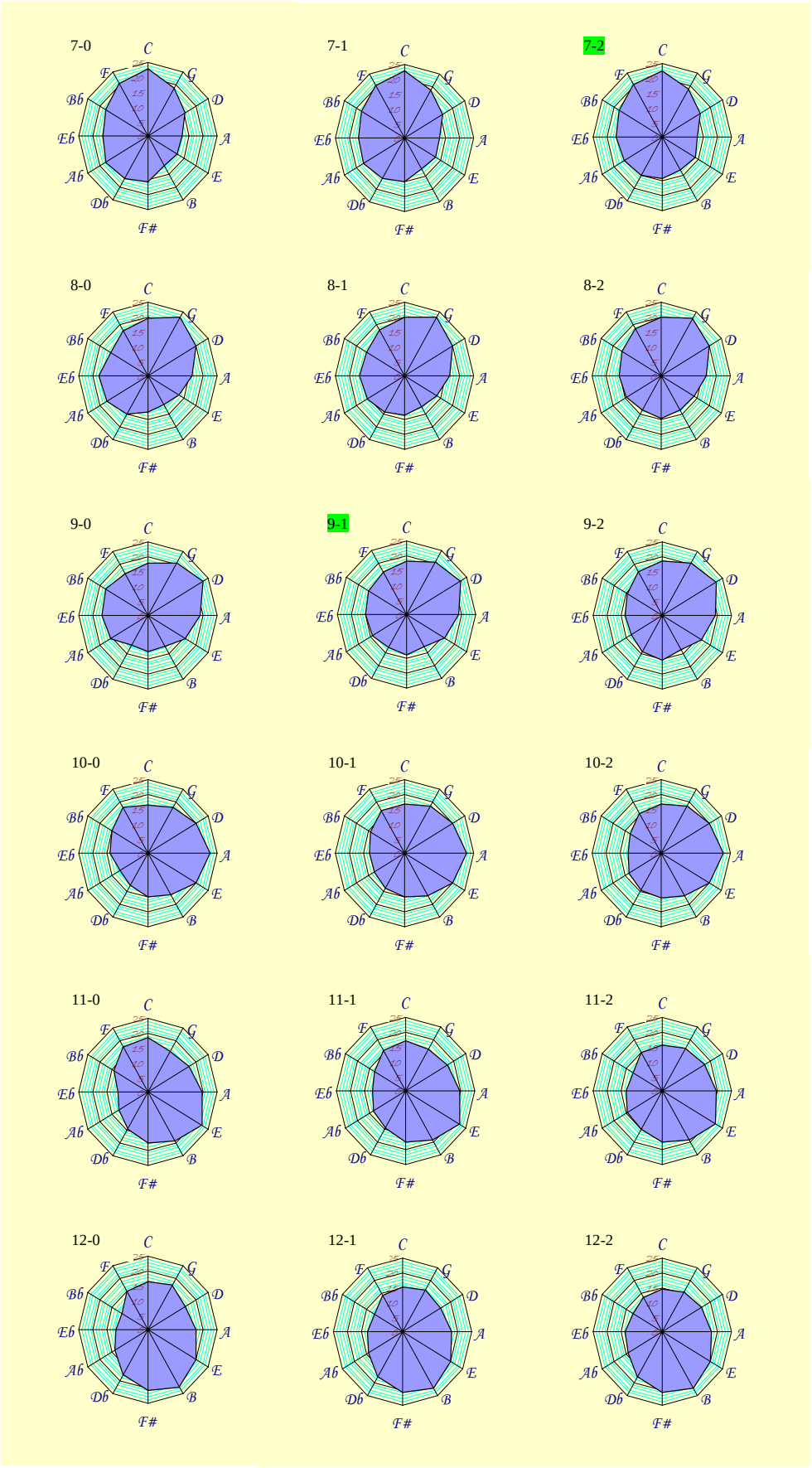


Figure 17: Euclidian distance in cents of the pure minor tetrachord from each tetrachord in Temperaments 7-x to 12-x for end beat-rates of 0, 1 and 2

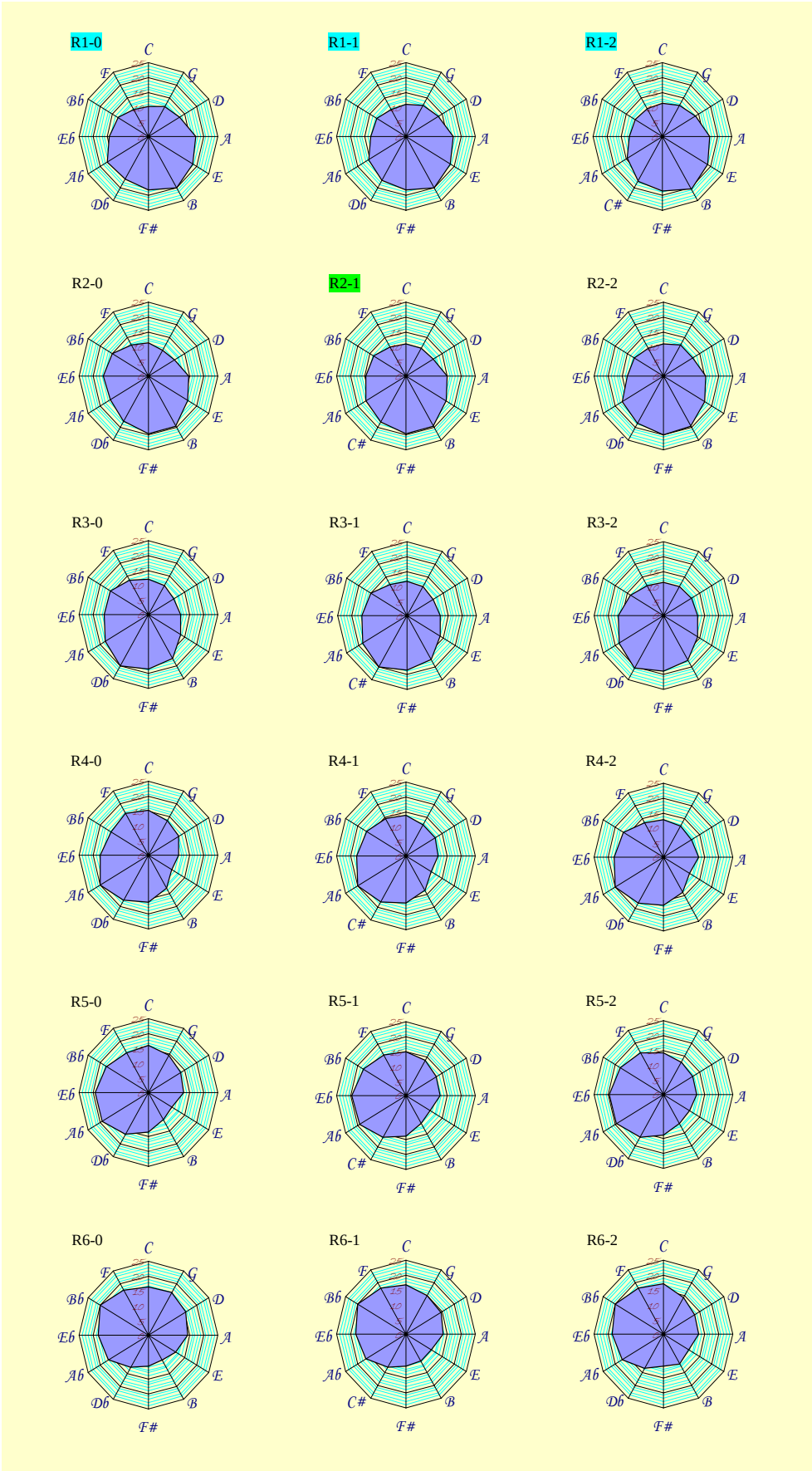


Figure 18: Euclidian distance in cents from the pure major tetrachord of each tetrachord in Temperaments R1-x to R6-x for end beat-rates of 0, 1 and 2

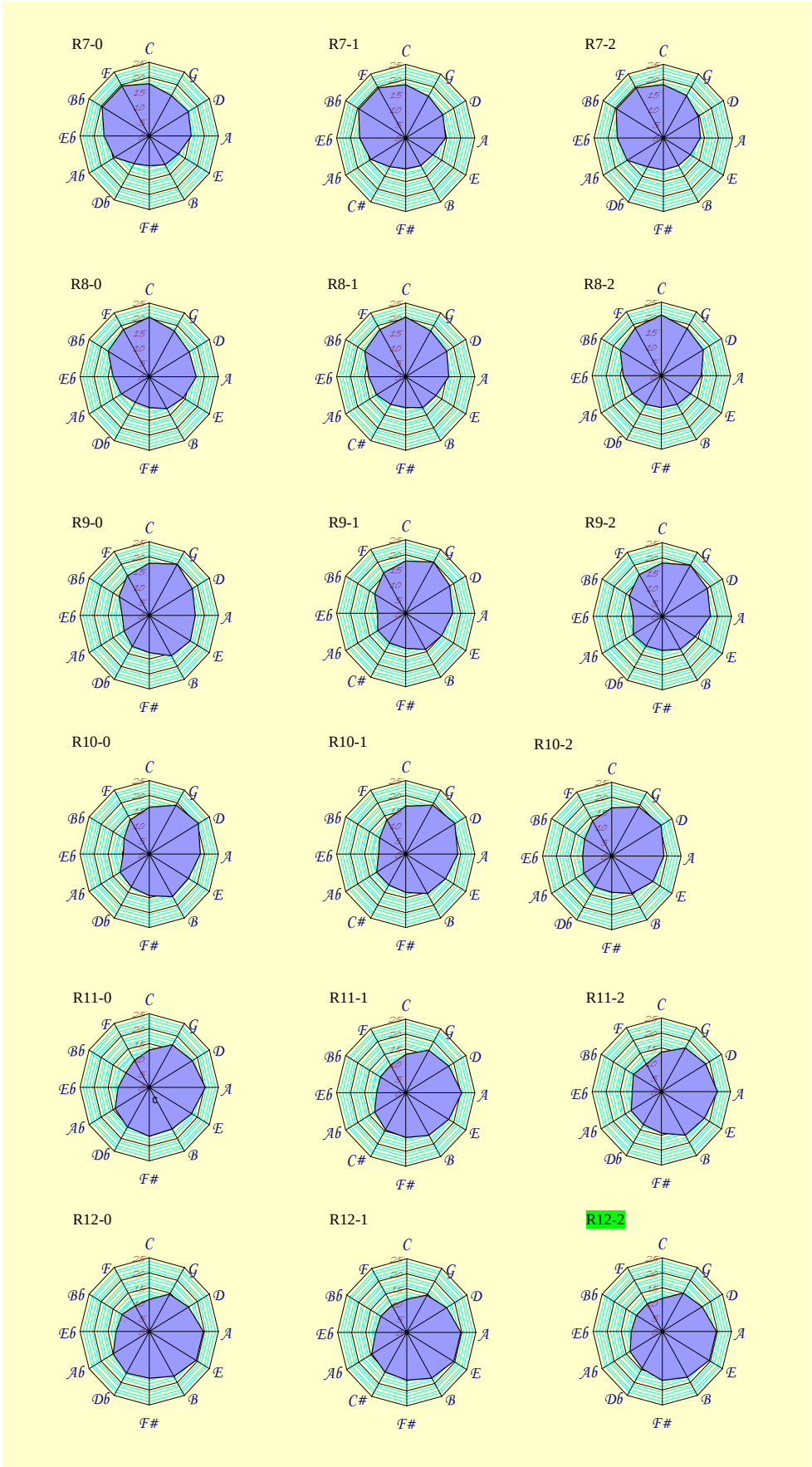


Figure 19: Euclidian distance in cents from the pure major tetrachord of each tetrachord in Temperaments R7-x to R12-x for end beat-rates of 0, 1 and 2

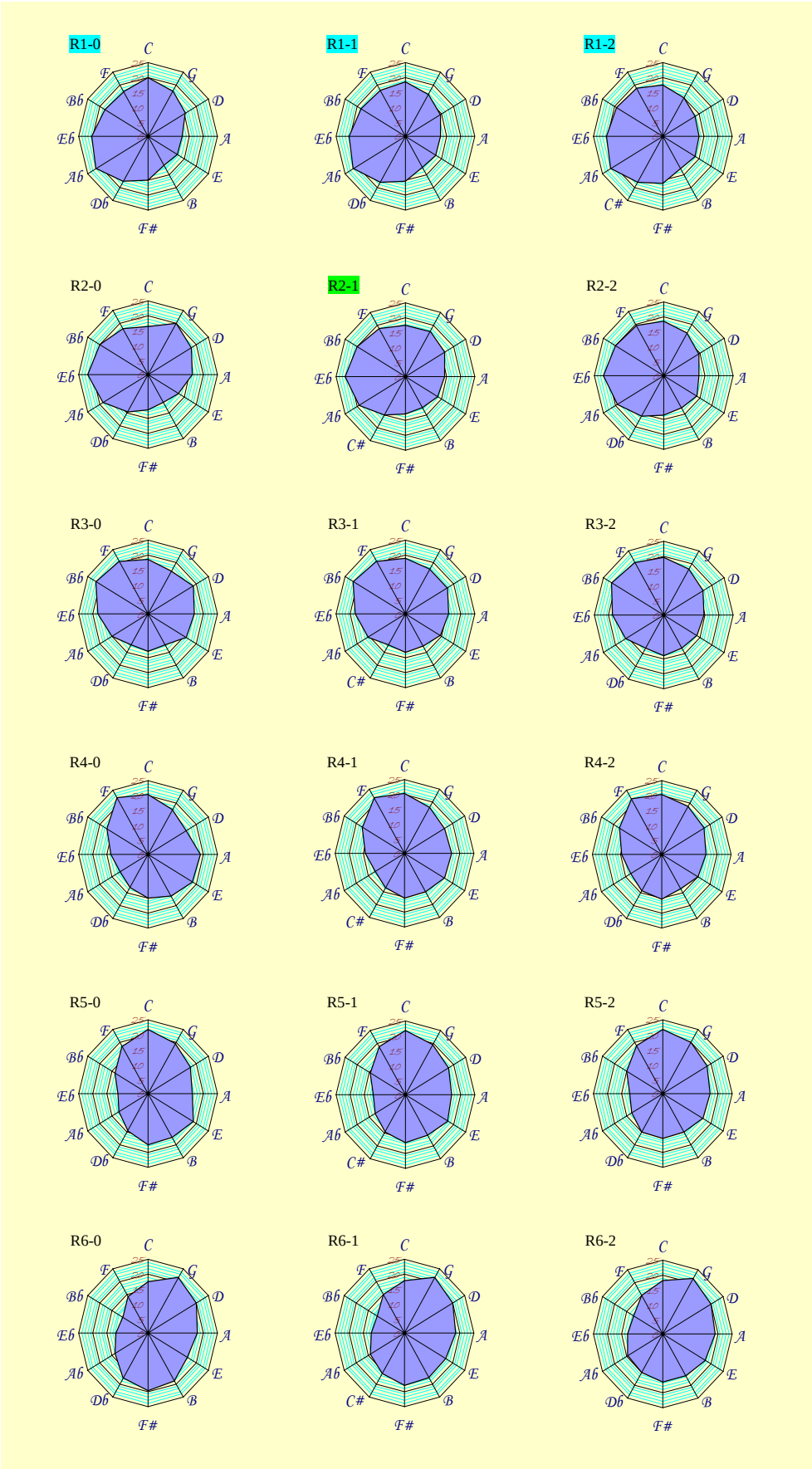


Figure 20: Euclidian distance in cents of the pure minor tetrachord from each tetrachord in Temperaments R1-x to R6-x for end beat-rates of 0, 1 and 2

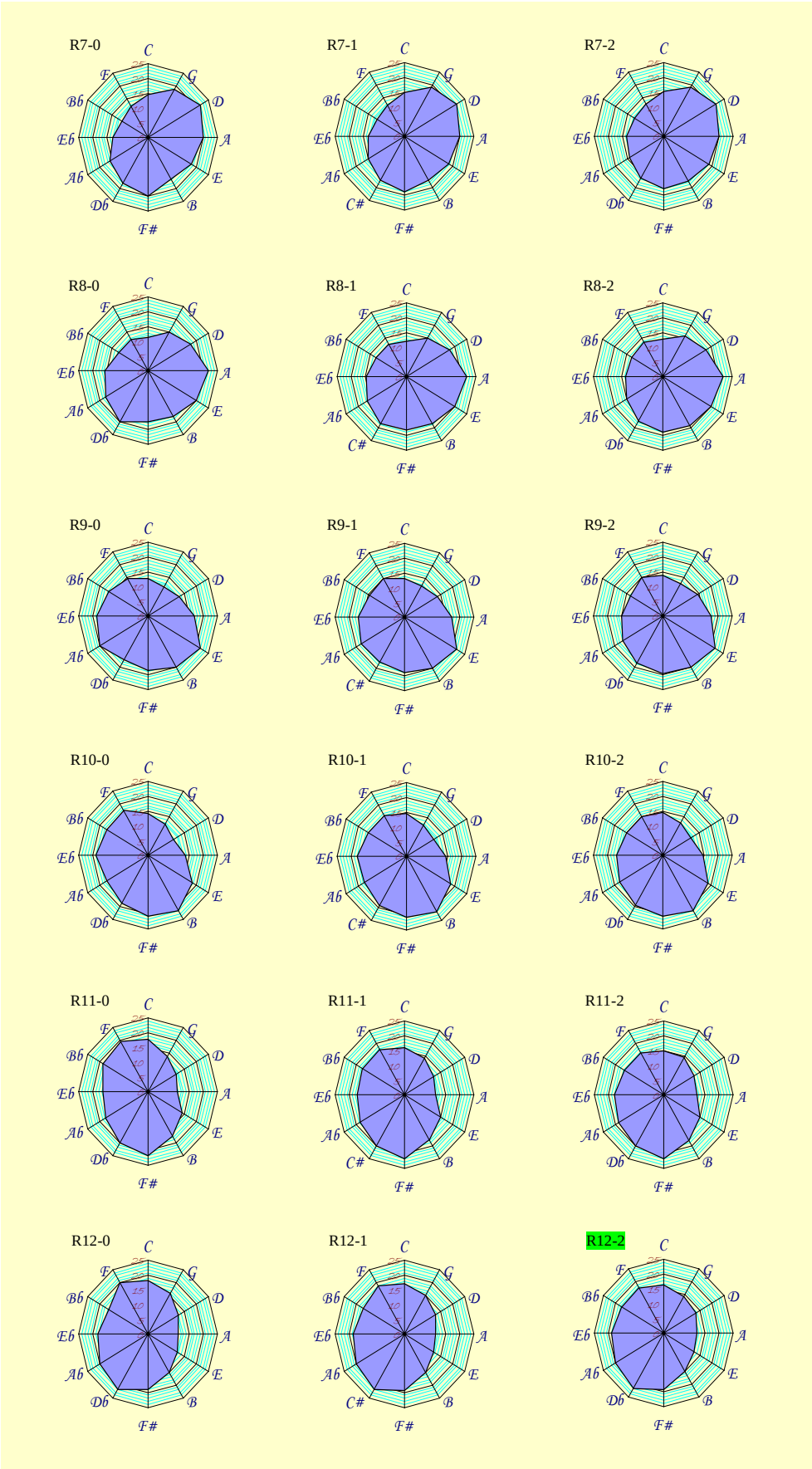


Figure 21: Euclidian distance in cents of the pure minor tetrachord from each tetrachord in Temperaments R7-x to R12-x for end beat-rates of 0, 1 and 2

Root	Minor 3rd	Major 3rd	Fifths
C	297	401	702
G	295	397	702
D	294	395	698
A	298	396	699
E	301	396	698
B	305	398	699
F#	303	399	699
Db	304	401	700
Ab	302	403	700
Eb	301	405	700
Bb	300	407	701
F	299	404	702

Table 28: Temperament 9-1 (Cammerton) thirds and fifths (cents). The best major third is D:F#.

Root	Minor 3rd	Major 3rd	Fifths
C	294	395	698
G	298	396	699
D	301	396	698
A	305	398	699
E	303	399	699
B	304	401	700
F#	302	403	700
Db	301	405	700
Ab	300	407	701
Eb	299	404	702
Bb	297	401	702
F	295	397	702

Table 29: Temperament 7-2 (Cornet-ton) thirds and fifths (cents). The best major third is C:E.

Root	Minor 3rd	Major 3rd	Fifths
C	299	396	700
G	299	394	699
D	301	395	698
A	304	399	699
E	304	402	698
B	305	405	699
F#	303	406	702
Db	300	405	702
Ab	297	403	702
Eb	294	401	700
Bb	296	399	701
F	297	397	700

Table 30: Temperament R2-1 (Cammerton) thirds and fifths (cents). Best major third is G:B.

Root	Minor 3rd	Major 3rd	Fifths
C	301	395	698
G	304	399	699
D	304	402	698
A	305	405	699
E	303	406	702
B	300	405	702
F#	297	403	702
Db	294	401	700
Ab	296	399	701
Eb	297	397	700
Bb	299	396	700
F	299	394	699

Table 31: Temperament 12-2 (Cornet-ton) thirds and fifths (cents). The best major third is F:A.

Root	Minor 3rd	Major 3rd	Fifths
C	297	393	697
G	300	394	699
D	302	397	698
A	306	401	699
E	304	404	698
B	305	406	702
F#	301	404	702
Db	298	403	702
Ab	294	403	700
Eb	296	400	700
Bb	298	399	701
F	299	396	702

Table 32: Temperament R1-0 thirds and fifths (cents)

	C	Db	D	Eb	E	F	F#	G	Ab	A	Bb	B	C
C	0	1103	996	903	799	702	601	498	403	298	203	101	0
Db	97	0	1093	1000	896	799	699	595	500	396	300	198	97
D	204	107	0	1106	1003	906	805	702	607	502	407	305	204
Eb	297	200	94	0	1097	999	899	795	700	596	500	398	297
E	401	304	197	103	0	1103	1002	899	804	699	603	502	401
F	498	401	294	201	97	0	1099	996	901	796	701	599	498
F#	599	501	395	301	198	101	0	1097	1002	897	801	699	599
G	702	605	498	405	301	204	103	0	1105	1000	905	803	702
Ab	797	700	593	500	396	299	198	95	0	1095	1000	898	797
A	902	804	698	604	501	404	303	200	105	0	1104	1003	902
Bb	997	900	793	700	597	499	399	295	200	96	0	1098	997
B	1099	1002	895	802	698	601	501	397	302	197	102	0	1099
C	1200	1103	996	903	799	702	601	498	403	298	203	101	0

Table 33: temperament 9-1 (Cammerton) all intervals (cents)

	C	Db	D	Eb	E	F	F#	G	Ab	A	Bb	B	C
C	0	1106	1003	906	805	702	607	502	407	305	204	107	0
Db	94	0	1097	999	899	795	700	596	500	398	297	200	94
D	197	103	0	1103	1002	899	804	699	603	502	401	304	197
Eb	294	201	97	0	1099	996	901	796	701	599	498	401	294
E	395	301	198	101	0	1097	1002	897	801	699	599	501	395
F	498	405	301	204	103	0	1105	1000	905	803	702	605	498
F#	593	500	396	299	198	95	0	1095	1000	898	797	700	593
G	698	604	501	404	303	200	105	0	1104	1003	902	804	698
Ab	793	700	597	499	399	295	200	96	0	1098	997	900	793
A	895	802	698	601	501	397	302	197	102	0	1099	1002	895
Bb	996	903	799	702	601	498	403	298	203	101	0	1103	996
B	1093	1000	896	799	699	595	500	396	300	198	97	0	1093
C	1200	1106	1003	906	805	702	607	502	407	305	204	107	0

Table 34: Temperament 7-2 (Cornet-ton) all intervals (cents)

	C	Db	D	Eb	E	F	F#	G	Ab	A	Bb	B	C
C	0	1105	1002	901	804	700	607	500	403	304	201	106	0
Db	95	0	1097	996	900	795	702	596	498	399	296	201	95
D	198	103	0	1099	1003	899	805	699	601	502	399	305	198
Eb	299	204	101	0	1104	999	906	799	702	603	500	405	299
E	396	300	197	96	0	1096	1002	896	798	699	596	502	396
F	500	405	301	201	104	0	1107	1000	903	803	701	606	500
F#	593	498	395	294	198	93	0	1094	996	897	794	699	593
G	700	604	501	401	304	200	106	0	1102	1003	901	806	700
Ab	797	702	599	498	402	297	204	98	0	1101	998	903	797
A	896	801	698	597	501	397	303	197	99	0	1097	1003	896
Bb	999	904	801	700	604	499	406	299	202	103	0	1105	999
B	1094	999	895	795	698	594	501	394	297	197	95	0	1094
C	1200	1105	1002	901	804	700	607	500	403	304	201	106	0

Table 35: Temperament R2-1 (Cammerton) all intervals (cents)

	C	Db	D	Eb	E	F	F#	G	Ab	A	Bb	B	C
C	0	1099	1003	899	805	699	601	502	399	305	199	103	0
Db	101	0	1104	999	906	799	702	603	500	405	299	204	101
D	197	96	0	1096	1002	896	798	699	596	502	396	300	197
Eb	301	201	104	0	1107	1000	903	803	701	606	500	405	301
E	395	294	198	93	0	1094	996	897	794	699	593	498	395
F	501	401	304	200	106	0	1102	1003	901	806	700	604	501
F#	599	498	402	297	204	98	0	1101	998	903	797	702	599
G	698	597	501	397	303	197	99	0	1097	1003	896	801	698
Ab	801	700	604	499	406	299	202	103	0	1105	999	904	801
A	895	795	698	594	501	394	297	197	95	0	1094	999	895
Bb	1002	901	804	700	607	500	403	304	201	106	0	1105	1002
B	1097	996	900	795	702	596	498	399	296	201	95	0	1097
C	1200	1099	1003	899	805	699	601	502	399	305	199	103	0

Table 36: Temperament R12-2 (Cornet-ton) all intervals (cents)

	C	Db	D	Eb	E	F	F#	G	Ab	A	Bb	B	C
C	0	1105	1004	903	807	702	607	503	403	306	203	109	0
Db	95	0	1099	998	902	797	702	598	498	401	298	204	95
D	196	101	0	1099	1003	898	803	699	599	502	399	305	196
Eb	297	202	101	0	1105	999	904	800	700	604	500	406	297
E	393	298	197	95	0	1095	1000	896	796	699	596	502	393
F	498	403	302	201	105	0	1105	1001	901	804	701	607	498
F#	593	498	397	296	200	95	0	1096	996	899	796	702	593
G	697	602	501	400	304	199	104	0	1100	1003	900	806	697
Ab	797	702	601	500	404	299	204	100	0	1103	1000	906	797
A	894	799	698	596	501	396	301	197	97	0	1096	1003	894
Bb	997	902	801	700	604	499	404	300	200	104	0	1106	997
B	1091	996	895	794	698	593	498	394	294	197	94	0	1091
C	1200	1105	1004	903	807	702	607	503	403	306	203	109	0

Table 37: Temperament R1-0 all intervals (cents)

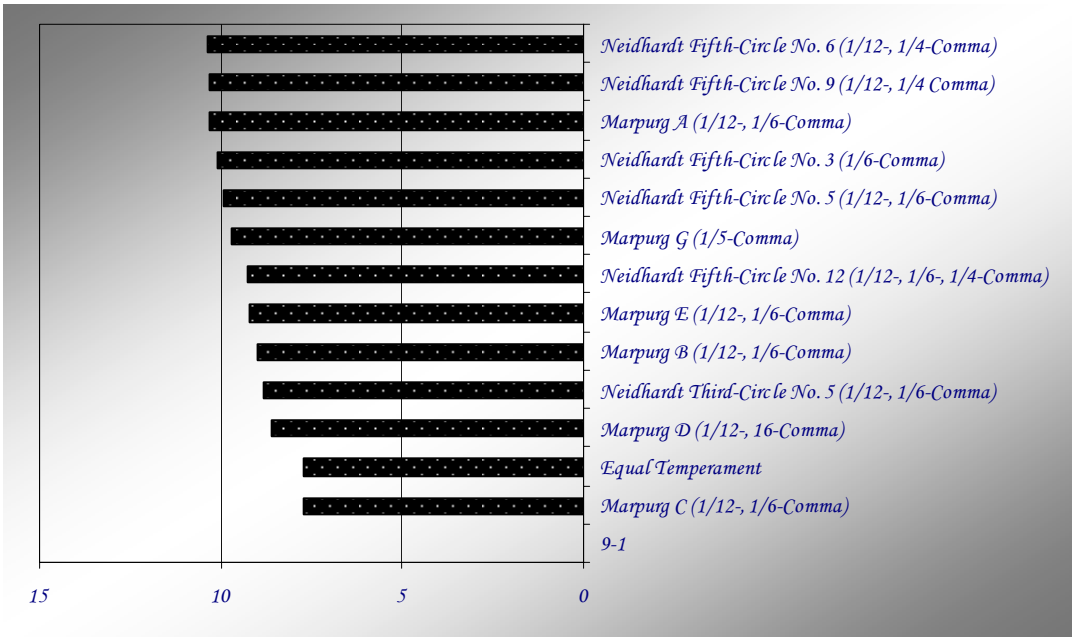


Figure 22: Temperament 9-1 (Cammerton) Euclidian distance to other temperaments (cents)

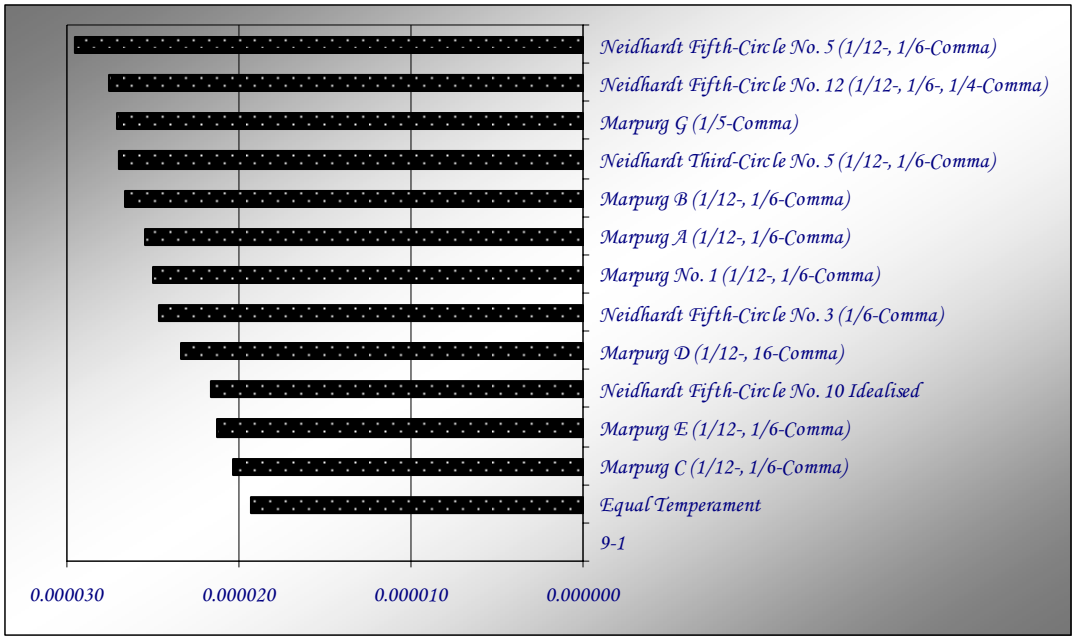


Figure 23: Temperament 9-1 (Cammerton) correlation distance to other temperaments

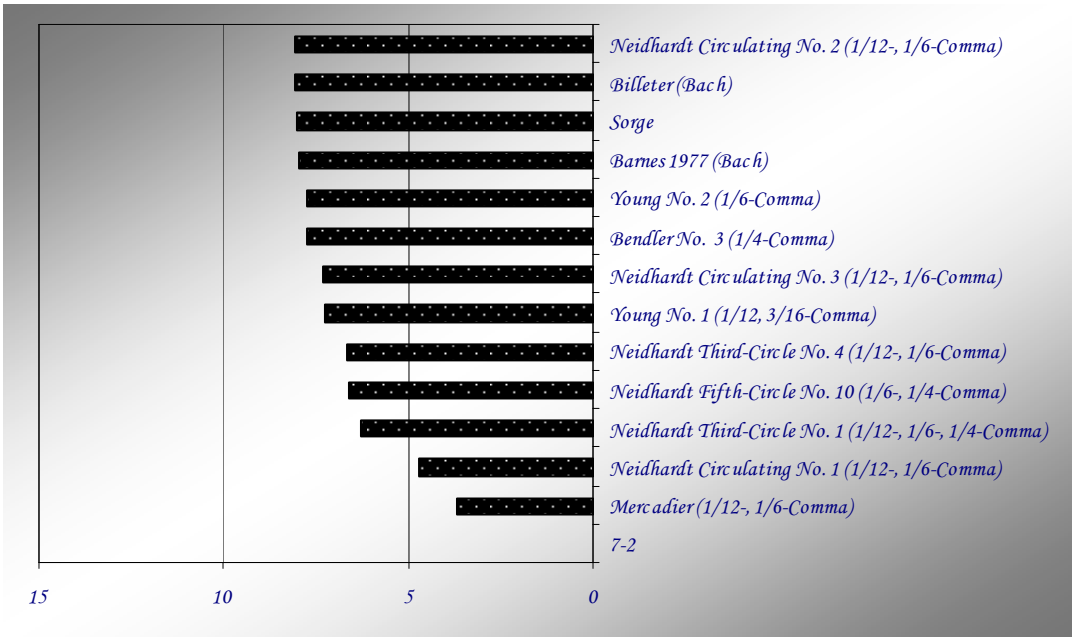


Figure 24: Temperament 7-2 (Cornet-ton) Euclidian distance to other temperaments (cents)

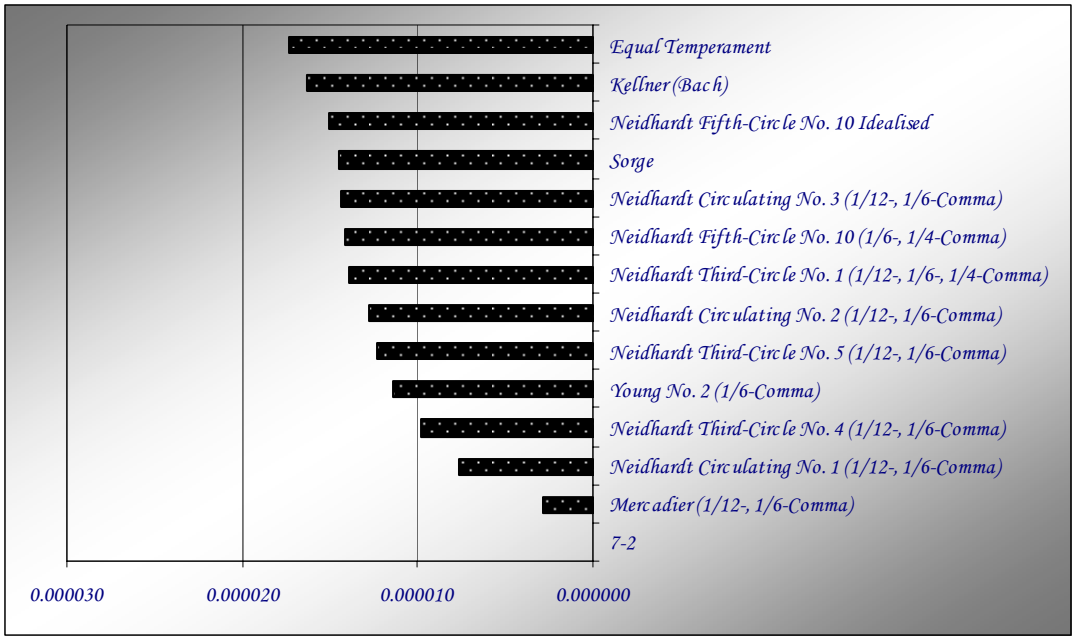


Figure 25: Temperament 7-2 (Cornet-ton) correlation distance to other temperaments

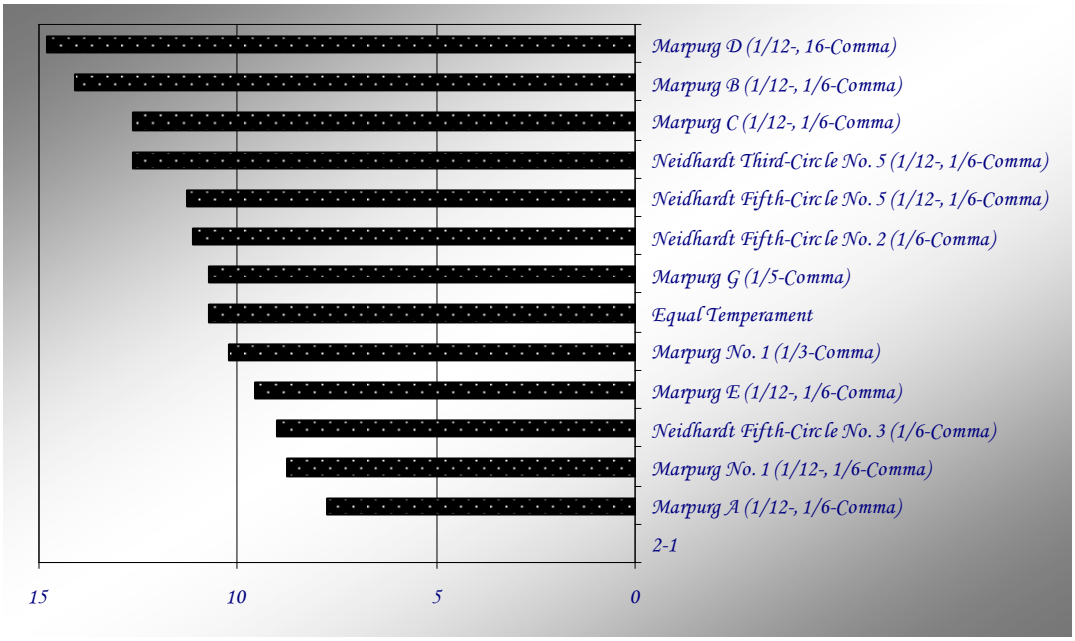


Figure 26: Temperament 2-1 (Cammerton) Euclidian distance to other temperaments (cents)

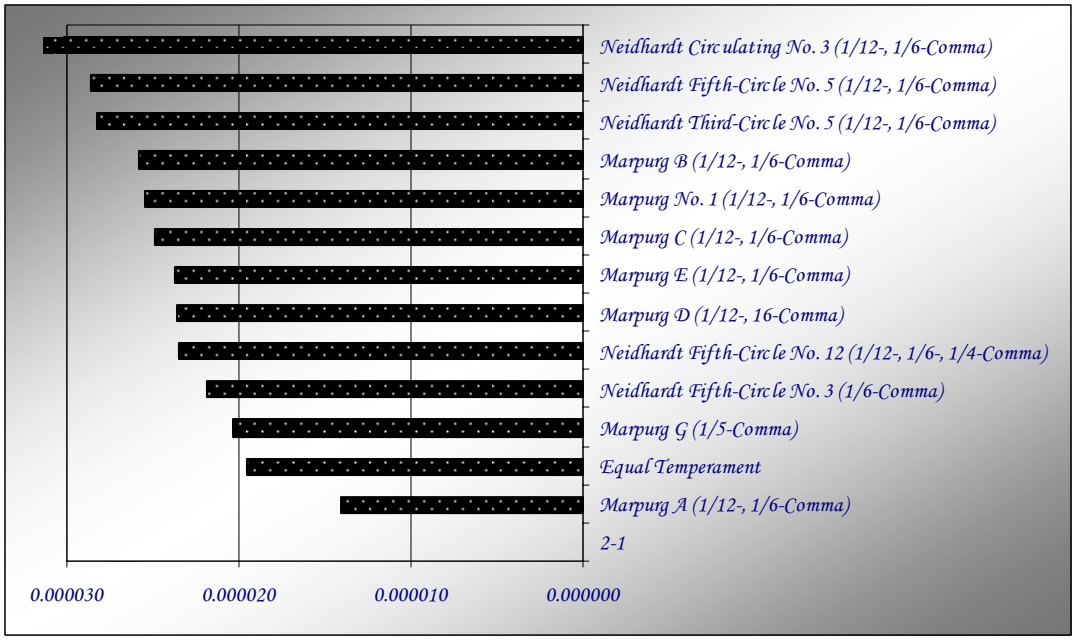


Figure 27: Temperament 2-1 (Cammerton) correlation distance to other temperaments

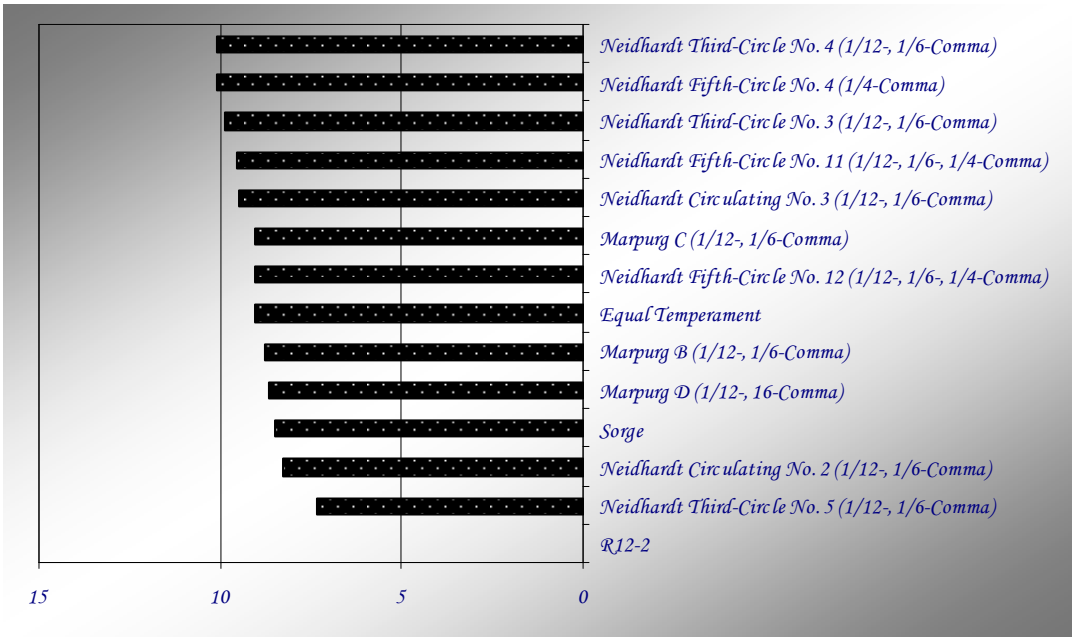


Figure 28: Temperament R12-2 (Cornet-ton) Euclidian distance to other temperaments (cents)

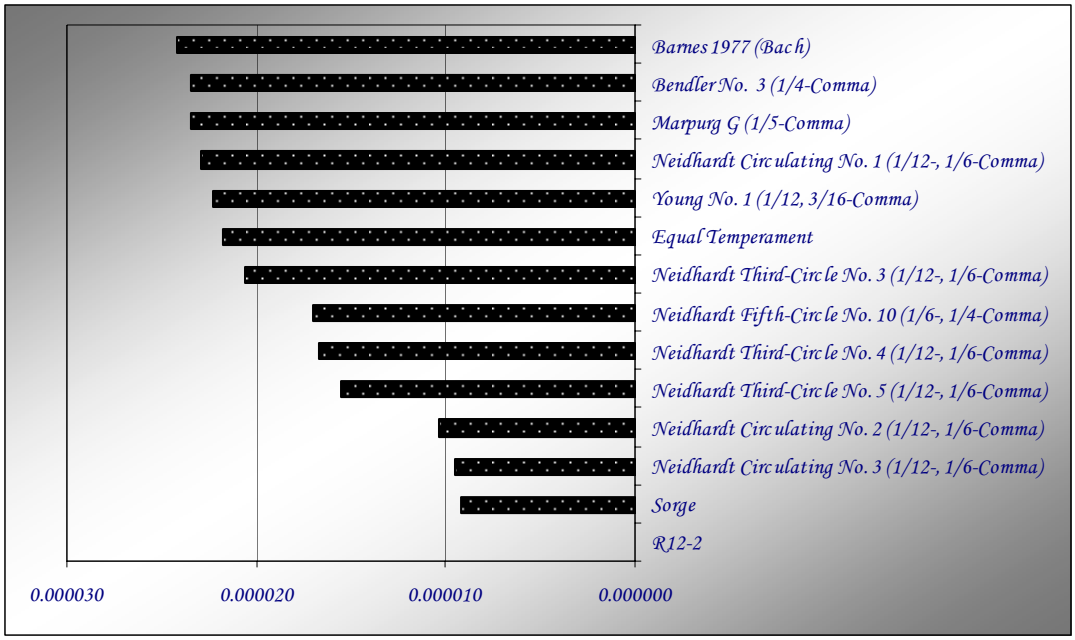


Figure 29: Temperament R12-2 (Cornet-ton) correlation distance to other temperaments

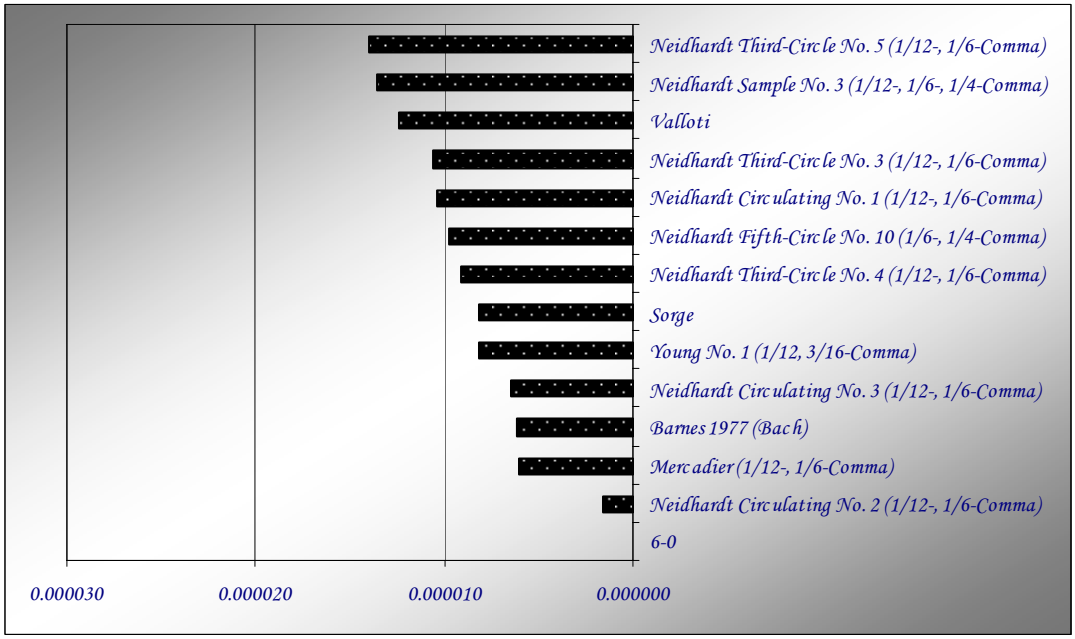


Figure 30: Temperament 6-0 correlation distance to other temperaments

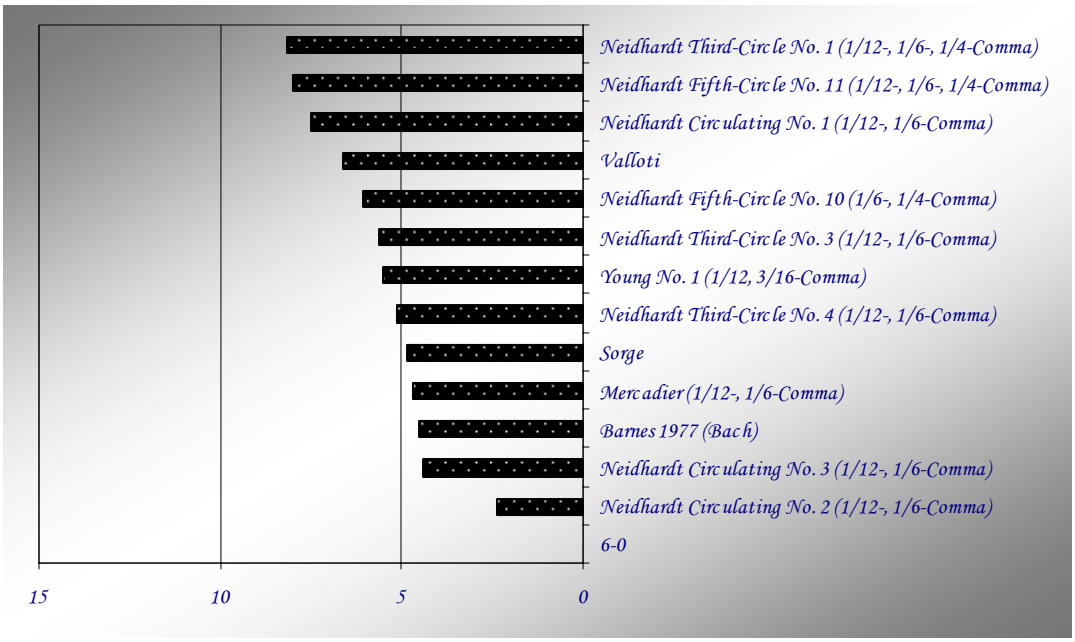


Figure 31: Temperament 6-0 Euclidian distance to other temperaments (cents)

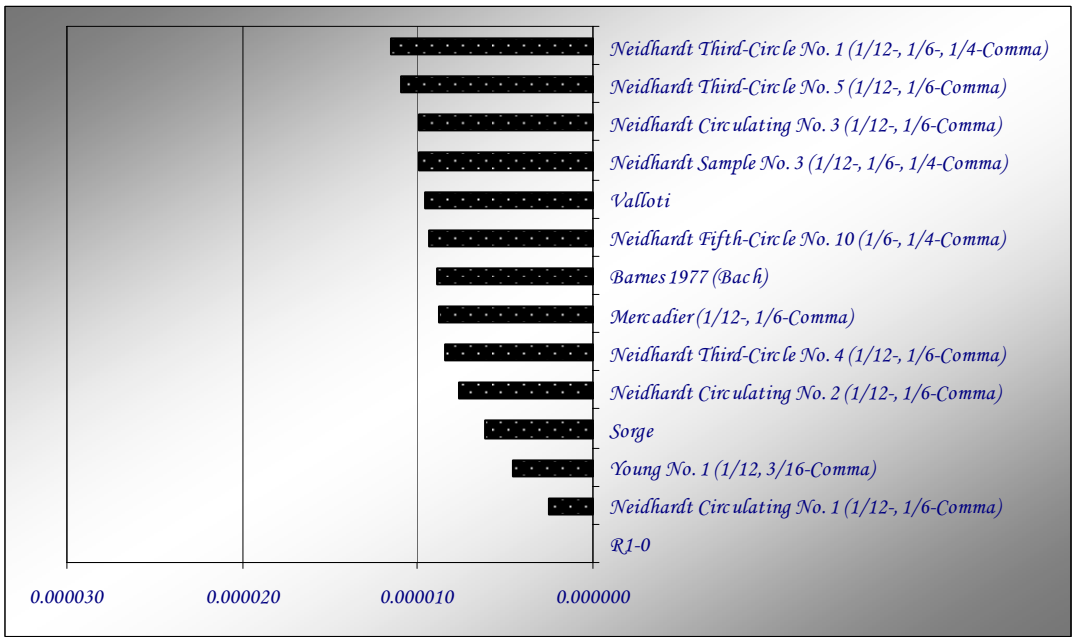


Figure 32: Temperament R1-0 correlation distance to other temperaments

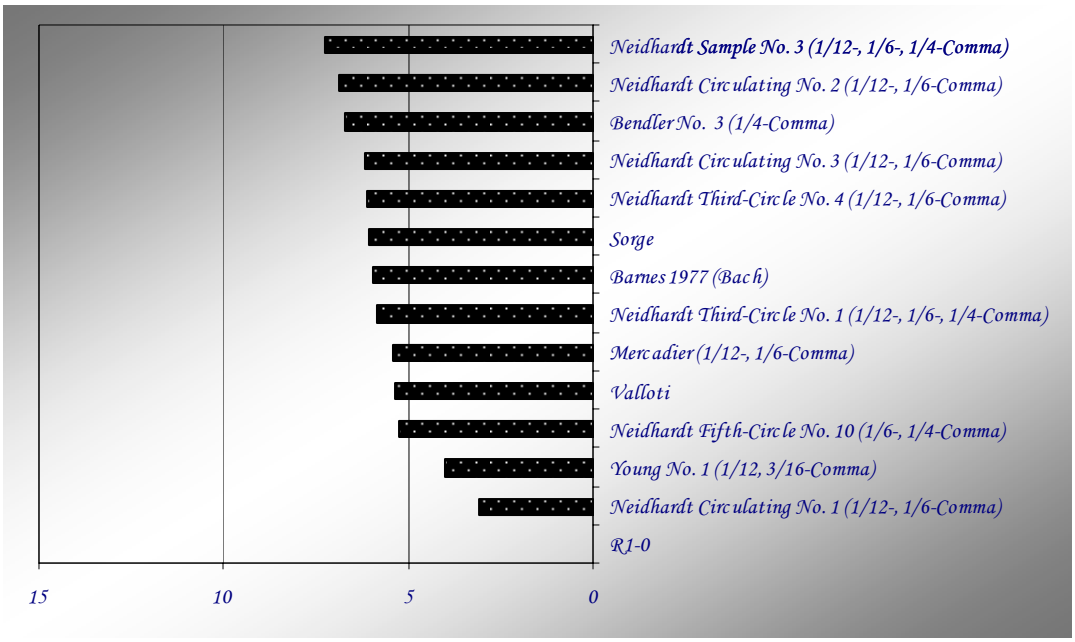


Figure 33: Temperament R1-0 Euclidian distance to other temperaments (cents)

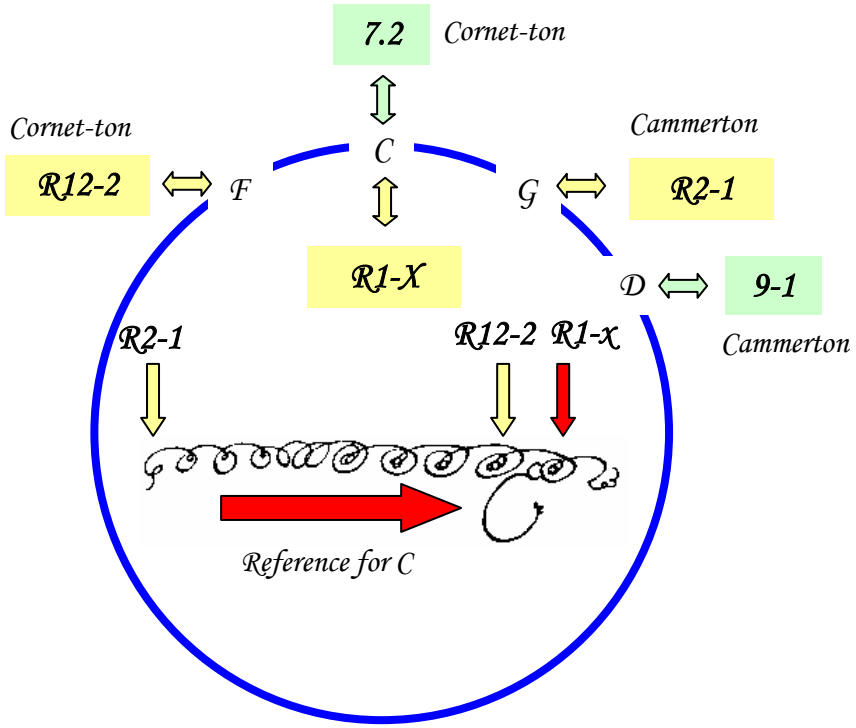


Figure 34: a circle of fifths with the location of the best thirds for two considered Cammerton-Cornet-ton solutions. Temperament 7-2 is Cornet-ton with the best third in *C*, while its pair Temperament 9-1 has its best third in *D*, two sharps clockwise of *C*. Cornet-ton Temperament R12-2 has the best third in *F*, one flat anticlockwise of *C*, while its Cammerton partner, Temperament R2-1, has the best third in *G*, one sharp clockwise of *C*. On the glyph Bach has marked a C reference point at the midpoint of R2-1 and R12-2 (to visualise this, keep in mind the glyph is a circle). The midpoint corresponds to R1-x, but of the three temperaments R1-0 (*a* = 410.376 Hz), R1-1 (*a*=441.286 Hz) and R1-2 (*a*=472.211 Hz), only R1-0 is existent at historical pitch standards.

	No. Clavier Movements	No. Organ Movements	No. Both
No sharps / flats	105	111	216
1-sharp	97	84	181
2-sharps	60	32	92
3-sharps	30	17	47
4-sharps	23	6	29
5-sharps / 7-flats	9	0	9
6-sharps / 5-flats	9	0	9
7-sharps / 5-flats	10	0	10
4-flats	10	0	10
3-flats	45	38	83
2-flats	61	41	102
1-flat	77	59	136

Figure 35: the frequency of sharps/flats occurrence in Bach's Clavier and Organ works derived from [5]. The correlation between the Clavier and Organ statistics is 0.994.

	Clavier	Organ	Both
R1-0	-0.97	-0.94	-0.96
R2-1 (Cammerton)	-0.91	-0.82	-0.87
7-2 (Cornet-ton)	-0.78	-0.72	-0.76
R12-2 (Cornet-ton)	-0.67	-0.69	-0.68
9-1 (Cammerton)	0.11	0.15	0.13

Figure 36: the correlations between temperaments derived from the glyph and the frequency of sharps/flats occurrence in Bach's Clavier and Organ works.

	Klavier	Organ	Both
Kirnberger No 3	-0.97	-0.94	-0.96
Neidhardt Sample No. 3 (1/12-, 1/6-, 1/4-Comma)	-0.96	-0.95	-0.96
Neidhardt Sample No. 2 (1/12-, 1/6-, 1/4-Comma)	-0.95	-0.94	-0.95
Neidhardt Circulating No. 1 (1/12-, 1/6-Comma)	-0.96	-0.93	-0.95
Young No. 1 (1/12, 3/16-Comma)	-0.96	-0.90	-0.94
Kelletats 1966 (Bach)	-0.95	-0.91	-0.93
Valloti	-0.96	-0.89	-0.93
Neidhardt Third-Circle No. 1 (1/12-, 1/6-, 1/4-Comma)	-0.93	-0.89	-0.92
Sorge	-0.90	-0.92	-0.91
Klais (Bach)	-0.93	-0.88	-0.91
Barnes 1977 (Bach)	-0.92	-0.86	-0.90
Kellner (Bach)	-0.91	-0.86	-0.89
Neidhardt Third-Circle No. 3 (1/12-, 1/6-Comma)	-0.90	-0.87	-0.89
Neidhardt Circulating No. 2 (1/12-, 1/6-Comma)	-0.90	-0.87	-0.89
Werckmeister Correct No. 1 (1/4-Comma)	-0.90	-0.85	-0.88
Mercadier (1/12-, 1/6-Comma)	-0.89	-0.83	-0.87
Neidhardt Third-Circle No. 4 (1/12-, 1/6-Comma)	-0.89	-0.83	-0.87
Neidhardt Fifth-Circle No. 10 (1/6-, 1/4-Comma)	-0.83	-0.86	-0.85
Neidhardt Third-Circle No. 5 (1/12-, 1/6-Comma)	-0.87	-0.81	-0.85
Billeter (Bach)	-0.88	-0.81	-0.85
Bendler No. 3 (1/4-Comma)	-0.85	-0.83	-0.85
Neidhardt Circulating No. 3 (1/12-, 1/6-Comma)	-0.83	-0.85	-0.85
Schlick	-0.87	-0.80	-0.84
Young No. 2 (1/6-Comma)	-0.84	-0.76	-0.81
Bendler No. 1 (1/3-Comma)	-0.75	-0.71	-0.73
Werckmeister Correct No. 2 (1/3-Comma)	-0.76	-0.66	-0.72
Bendler No. 2 (1/3-Comma)	-0.70	-0.70	-0.70
Silbermann (1/6-Comma)	-0.74	-0.65	-0.70
Aron's Meantone (1/4-Comma)	-0.74	-0.65	-0.70
Werckmeister Correct No. 3 (1/4-Comma)	-0.64	-0.57	-0.61
Neidhardt Fifth-Circle No. 11 (1/12-, 1/6-, 1/4-Comma)	-0.53	-0.57	-0.56
Neidhardt Fifth-Circle No. 10 Idealised	-0.38	-0.39	-0.39
Neidhardt Fifth-Circle No. 12 (1/12-, 1/6-, 1/4-Comma)	-0.20	-0.35	-0.28
Marpurg D (1/12-, 16-Comma)	-0.10	-0.22	-0.16
Neidhardt Fifth-Circle No. 4 (1/4-Comma)	-0.05	-0.18	-0.12
Neidhardt Fifth-Circle No. 6 (1/12-, 1/4-Comma)	-0.11	-0.12	-0.11
Marpurg G (1/5-Comma)	0.03	-0.08	-0.03
Neidhardt Fifth-Circle No. 7 (1/6, 1/4 Comma)	0.01	-0.02	0.00
Equal Temperament	0.00	0.00	0.00
Marpurg C (1/12-, 1/6-Comma)	0.00	0.00	0.00
Marpurg No. 1 (1/12-, 1/6-Comma)	0.00	0.00	0.00
Marpurg No. 2 (1/12-. 5/24-Comma)	0.00	0.00	0.00
Neidhardt Fifth-Circle No. 2 (1/6-Comma)	0.00	0.00	0.00
Neidhardt Fifth-Circle No. 3 (1/6-Comma)	0.00	0.00	0.00
Neidhardt Fifth-Circle No. 9 (1/12-, 1/4 Comma)	0.00	0.00	0.00
Marpurg B (1/12-, 1/6-Comma)	0.02	0.03	0.03
Marpurg E (1/12-, 1/6-Comma)	0.09	0.14	0.12
Neidhardt Fifth-Circle No. 5 (1/12-, 1/6-Comma)	0.09	0.17	0.13
Bermudo Vihuela (1/6-, 1/2-Comma)	0.32	0.31	0.32
Pythagorean	0.74	0.65	0.70

Figure 37: correlations between historical temperaments and the frequency of sharps/flats occurrence in Bach's Clavier and Organ works.

	C	Db	D	Eb	E	F	F#	G	Ab	A	Bb	B
1-0	0	104.30	199.97	300.77	401.77	498.05	605.68	699.59	801.80	899.81	998.75	1103.72
2-0	0	108.42	202.28	304.65	402.56	501.71	606.47	701.96	805.77	902.06	1002.50	1104.51
3-0	0	103.47	199.14	304.39	399.50	501.62	601.52	697.18	805.43	898.96	1002.34	1099.56
4-0	0	100.77	195.71	304.68	398.16	501.72	598.82	697.04	802.73	897.67	1002.52	1098.17
5-0	0	96.51	195.90	300.42	395.52	501.64	596.25	697.15	798.46	893.57	1002.37	1095.57
6-0	0	95.96	196.08	297.65	393.00	501.56	595.66	697.26	795.69	893.84	999.60	1094.95
7-0	0	93.86	196.23	294.14	393.29	498.05	593.53	697.35	793.63	894.08	996.09	1091.58
8-0	0	96.28	200.57	296.24	397.05	498.05	594.32	701.96	795.87	898.07	996.09	1095.02
9-0	0	95.49	203.91	297.77	400.14	498.05	597.20	701.96	797.44	901.26	997.54	1097.99
10-0	0	100.44	203.91	299.57	404.82	499.94	602.06	701.96	797.62	905.87	999.40	1102.77
11-0	0	101.22	201.44	297.37	405.35	499.89	602.75	699.49	798.52	903.40	999.32	1103.40
12-0	0	103.70	199.88	300.01	403.79	499.86	605.15	699.54	801.10	901.84	997.90	1105.75
R1-0	0	95.02	196.04	297.45	392.92	498.05	593.07	697.24	796.98	893.78	997.42	1091.11
R2-0	0	95.76	200.48	299.67	396.73	499.97	593.81	701.96	797.72	897.84	999.46	1094.59
R3-0	0	93.51	201.40	297.42	397.86	499.92	595.12	699.45	795.47	898.89	999.38	1095.82
R4-0	0	96.71	199.82	295.97	400.81	499.88	598.23	699.50	794.02	901.77	997.93	1098.87
R5-0	0	97.80	199.91	294.14	401.68	498.05	599.24	699.56	795.21	899.73	996.09	1099.83
R6-0	0	102.22	202.26	298.32	402.48	498.05	603.77	701.96	799.49	902.00	996.09	1104.44
R7-0	0	104.71	203.91	300.97	404.21	498.05	606.19	701.96	802.08	903.70	998.83	1104.24
R8-0	0	108.89	203.91	304.92	406.33	501.81	606.93	701.96	806.12	905.87	1002.66	1106.30
R9-0	0	103.82	199.01	304.66	402.92	501.71	603.59	697.05	805.78	900.96	1002.50	1102.93
R10-0	0	102.75	195.93	304.41	399.84	501.63	602.48	697.17	802.45	897.88	1002.35	1101.79
R11-0	0	100.04	196.10	300.26	395.83	501.55	599.74	697.27	799.78	893.87	1002.22	1097.78
R12-0	0	95.02	196.04	297.45	392.92	498.05	593.07	697.24	796.98	893.78	997.42	1091.11

Table 38: Temperaments 1-0 to R12-0 in cents

	C	Db	D	Eb	E	F	F#	G	Ab	A	Bb	B
1-1	0	104.95	200.24	301.95	402.19	499.69	606.10	699.76	802.77	900.24	1000.21	1104.15
2-1	0	106.70	200.22	303.66	400.84	501.36	604.75	699.74	804.49	900.20	1001.89	1102.79
3-1	0	102.63	197.95	303.73	398.54	501.39	600.67	697.49	804.58	897.91	1001.94	1098.72
4-1	0	99.87	196.42	303.78	396.98	501.41	597.92	697.46	801.83	896.37	1001.97	1097.15
5-1	0	96.15	196.36	300.06	394.88	501.43	595.79	697.43	798.10	894.28	1002.01	1095.04
6-1	0	95.53	196.66	297.39	394.10	501.30	595.12	697.61	795.43	894.73	999.34	1094.32
7-1	0	93.68	196.61	294.14	394.01	498.05	593.27	697.58	793.56	894.66	996.09	1092.48
8-1	0	95.85	200.81	296.10	397.81	498.05	595.55	701.96	795.61	898.62	996.09	1096.06
9-1	0	97.21	203.91	297.43	400.87	498.05	598.57	701.96	796.95	901.70	997.41	1099.10
10-1	0	101.28	203.91	299.23	405.01	499.82	602.67	701.95	798.77	905.87	999.19	1103.22
11-1	0	102.24	201.64	299.01	405.55	499.74	603.48	699.69	799.91	903.60	999.06	1103.92
12-1	0	104.24	200.10	300.97	404.01	499.76	605.51	699.67	801.89	902.06	999.08	1105.97
R1-1	0	96.33	196.59	298.86	393.96	499.69	594.37	697.56	798.28	894.62	998.97	1092.42
R2-1	0	95.21	198.50	299.12	395.51	499.78	593.26	699.64	797.17	896.31	999.13	1093.77
R3-1	0	93.29	200.01	297.20	396.96	499.80	594.66	699.62	795.25	897.79	999.16	1095.19
R4-1	0	96.15	200.17	295.82	399.41	499.73	597.38	699.71	793.86	900.13	997.77	1097.80
R5-1	0	97.36	200.14	294.14	400.69	498.05	598.61	699.69	795.04	900.08	996.09	1099.05
R6-1	0	101.29	202.38	297.99	402.90	498.05	602.57	701.96	798.93	902.30	996.09	1103.04
R7-1	0	103.96	203.91	300.61	404.40	498.05	605.26	701.96	801.56	903.81	998.68	1104.53
R8-1	0	107.58	203.91	304.17	406.44	501.54	607.27	701.96	805.15	905.87	1002.20	1106.55
R9-1	0	104.71	199.45	303.72	403.36	501.38	604.32	697.49	804.56	901.40	1001.93	1103.55
R10-1	0	103.41	196.41	303.79	400.32	501.41	603.04	697.46	803.25	898.36	1001.97	1102.27
R11-1	0	101.09	196.70	301.61	396.75	501.28	600.66	697.63	801.01	894.80	1001.75	1098.71
R12-1	0	99.95	196.65	300.44	394.09	501.30	598.00	697.60	799.85	894.72	1000.57	1096.04

Table 39: Temperaments 1-1 to R12-1 in cents

	C	Db	D	Eb	E	F	F#	G	Ab	A	Bb	B
1-2	0	105.53	200.48	302.98	402.56	501.13	606.47	699.90	803.61	900.60	1001.48	1104.51
2-2	0	105.28	198.51	302.83	399.41	501.08	603.32	697.91	803.42	898.66	1001.39	1101.37
3-2	0	101.89	196.90	303.16	397.69	501.19	599.93	697.75	803.84	896.99	1001.59	1097.98
4-2	0	99.11	197.01	303.03	395.99	501.14	597.16	697.82	801.07	895.27	1001.51	1096.30
5-2	0	95.82	196.77	299.74	394.31	501.25	595.39	697.68	797.78	894.91	1001.69	1094.57
6-2	0	95.17	197.16	297.17	395.04	501.08	594.65	697.91	795.21	895.50	999.12	1093.77
7-2	0	93.52	196.96	294.14	394.66	498.05	593.04	697.79	793.50	895.19	996.09	1093.30
8-2	0	95.49	201.01	295.97	398.46	498.05	596.61	701.96	795.38	899.09	996.09	1096.96
9-2	0	98.63	203.91	297.14	401.47	498.05	599.71	701.96	796.54	902.06	997.29	1100.02
10-2	0	102.02	203.91	298.93	405.18	499.71	603.21	701.96	799.78	905.87	999.01	1103.61
11-2	0	103.09	201.81	300.41	405.72	499.62	604.10	699.86	801.10	903.77	998.84	1104.36
12-2	0	104.73	200.30	301.83	404.21	499.67	605.83	699.79	802.60	902.25	1000.15	1106.16
R1-2	0	97.46	197.06	300.08	394.87	501.12	595.51	697.85	799.42	895.36	1000.31	1093.55
R2-2	0	94.76	196.87	298.67	394.50	499.63	592.81	697.74	796.72	895.06	998.85	1093.10
R3-2	0	93.10	198.79	297.01	396.17	499.69	594.26	699.76	795.06	896.83	998.97	1094.63
R4-2	0	95.69	200.46	295.68	398.23	499.59	596.66	699.89	793.73	898.74	997.64	1096.90
R5-2	0	96.98	200.34	294.14	399.81	498.05	598.05	699.82	794.88	900.39	996.09	1098.36
R6-2	0	100.50	202.49	297.72	403.25	498.05	601.55	701.96	798.44	902.56	996.09	1101.84
R7-2	0	103.28	203.91	300.29	404.57	498.05	604.42	701.96	801.10	903.92	998.56	1104.79
R8-2	0	106.45	203.91	303.51	406.53	501.31	607.56	701.96	804.30	905.87	1001.80	1106.76
R9-2	0	105.45	199.82	302.93	403.73	501.11	604.93	697.86	803.55	901.77	1001.45	1104.06
R10-2	0	104.00	196.84	303.25	400.75	501.22	603.54	697.71	803.95	898.79	1001.64	1102.70
R11-2	0	102.00	197.22	302.76	397.55	501.05	601.46	697.94	802.06	895.59	1001.34	1099.50
R12-2	0	100.63	197.01	301.29	394.76	501.14	598.67	697.82	800.63	895.27	1001.50	1096.72

Table 40: Temperaments 1-2 to R12-2 in cents

7-2	C	Db	D	Eb	E	F	F#	G	Ab	A	Bb	B
C	0	93.52	196.96	294.14	394.66	498.05	593.04	697.79	793.50	895.19	996.09	1093.30
Db	0	106.70	200.22	303.66	400.84	501.36	604.75	699.74	804.49	900.20	1001.89	1102.79
D	0	97.21	203.91	297.43	400.87	498.05	598.57	701.96	796.95	901.70	997.41	1099.10
Eb	0	100.90	198.11	304.81	398.32	501.77	598.94	699.47	802.85	897.85	1002.59	1098.31
E	0	101.70	202.59	299.80	406.50	500.02	603.46	700.64	801.17	904.55	999.54	1104.29
F	0	95.71	197.41	298.30	395.51	502.21	595.73	699.17	796.35	896.88	1000.26	1095.25
F#	0	104.75	200.46	302.15	403.05	500.26	606.96	700.48	803.92	901.10	1001.62	1105.01
G	0	94.99	199.74	295.45	397.15	498.05	595.26	701.96	795.47	898.91	996.09	1096.62
Ab	0	103.38	198.38	303.12	398.83	500.53	601.43	698.64	805.34	898.85	1002.29	1099.47
A	0	100.53	203.91	298.90	403.65	499.36	601.06	701.96	799.17	905.87	999.38	1102.82
Bb	0	97.18	197.71	301.09	396.08	500.83	596.54	698.24	799.13	896.34	1003.04	1096.56
B	0	103.44	200.62	301.15	404.53	499.52	604.27	699.98	801.68	902.57	999.78	1106.48

Table 41: all transpositions of Temperament 7-2

R12-2	C	Db	D	Eb	E	F	F#	G	Ab	A	Bb	B
C	0.00	100.63	197.01	301.29	394.76	501.14	598.67	697.82	800.63	895.27	1001.50	1096.72
Db	0.00	103.28	203.91	300.29	404.57	498.04	604.42	701.95	801.10	903.91	998.55	1104.78
D	0.00	95.22	198.50	299.13	395.51	499.79	593.26	699.64	797.17	896.32	999.13	1093.77
Eb	0.00	106.23	201.45	304.73	405.36	501.74	606.01	699.49	805.87	903.40	1002.55	1105.36
E	0.00	94.64	200.87	296.09	399.37	500.00	596.38	700.65	794.13	900.51	998.04	1097.19
F	0.00	102.82	197.46	303.68	398.90	502.18	602.81	699.19	803.47	896.95	1003.32	1100.86
F#	0.00	99.15	201.96	296.60	402.83	498.05	601.33	701.96	798.34	902.61	996.09	1102.47
G	0.00	97.53	196.68	299.49	394.13	500.36	595.58	698.86	799.49	895.87	1000.15	1093.62
Ab	0.00	106.38	203.91	303.06	405.87	500.51	606.74	701.96	805.24	905.87	1002.25	1106.52
A	0.00	93.48	199.86	297.39	396.53	499.35	593.99	700.21	795.43	898.71	999.34	1095.72
Bb	0.00	104.28	197.75	304.13	401.66	500.81	603.62	698.26	804.49	899.71	1002.99	1103.62
B	0.00	96.38	200.66	294.14	400.51	498.05	597.19	700.01	794.65	900.87	996.09	1099.37

Table 42: all transpositions of Temperament 12-2

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Acknowledgements

I would like to express my gratitude to those who have supported me during the preparation of this article: the knowledgeable Thomas Braatz, living in the greater Chicago area, who provided historical background material and assisted with examples of Bach's handwriting; the staff at the Institut für Musikwissenschaft, Bern, who tracked down an elusive research article on my behalf; and last, but not least, my wife, Gabi, who has helped in many ways. My interest in Bach's tuning was stimulated by the late Herbert Anton Kellner, who kindly invited me to participate in his private tuning circle and provided me with several research articles. Andreas Sparschuh, a mathematician from the Technical University Darmstadt, first proposed the link between Bach's glyph and tuning, and I am grateful to Michael Zapf of Frankfurt for bringing this work to my attention.